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Exponentially improved asymptotic expansions for resonances of an elliptic cylinder

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Abstract. Scattering by an elliptic cylinder is considered. Asymptotic expansions for Regge poles and resonances are derived from the uniform asymptotic expansions of Mathieu functions and modified Mathieu functions constructed by applying the Langer–Olver method. In addition, asymptotic expansions for resonances are exponentially improved by emphasizing the role of the symmetries of the scatterer. The splitting up of resonances is then explained in terms of the symmetry breaking $O(2) \longrightarrow \mathcal{C}_{2v}$.

1. Introduction

In the elliptic cylinder geometry, the Helmholtz equation can be solved by separation of variables and its general solution then constructed in terms of Mathieu functions and modified Mathieu functions [1–3]. As a consequence, the problems of diffraction and scattering of waves by elliptic cylinders (see, for example, [4] for a review) can be solved exactly and therefore have attracted wide attention in the context of theoretical and technical applications. Indeed,

- the elliptic cylinder is an obvious generalization of the circular cylinder and a very particular but non-trivial case of the convex cylinder. Because its cross section can be modified by changing its axial ratio, it can be used to approximate a great variety of geometrical shapes;
- various approximation techniques (see [4] and references therein) can be tested by comparison with the exact theory. In particular, scattering has been studied with the help of the Keller geometrical theory of diffraction [5–8], developed in the context of a convex cylinder with arbitrary variable curvature. The solution thus obtained has been compared with the leading term of the asymptotic expansion of the exact solution [9];
- scattering can be used in connection with semiclassical quantization of billiards from the inside–outside duality (see [10] for a review) and, in particular, in order to obtain the spectrum of the quantum elliptic billiard.

Scattering by an elliptic cylinder is considered here by emphasizing the role of the symmetries of the scatterer. Symmetry considerations greatly simplify the mathematical analysis of a scattering problem. For example, in the case of a circular cylinder, the invariance of the Helmholtz equation under the continuous group $O(2)$ (i.e. under rotations about the

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cylinder axis) leads to the search for mode solutions of the form $f(\rho)\exp(\pm in\varphi)$ (here ρ and φ, respectively, denote the radial and angular coordinates of the polar coordinate system defined with respect to the cylinder axis; see figure 1). This is directly linked to the following mathematical result: the functions $\exp(\pm in\theta)$, with $n \in \mathbb{N}^*$ fixed, form a basis for a two-dimensional representation of $O(2)$. Consequently, resonances of the circular cylinder are twofold degenerate. In the elliptic cylinder case, the invariance of the circular cylinder under the continuous group $O(2)$ is broken. Even so, the system is invariant under four symmetry transformations (see figure 3):

(a) E, the identity transformation;
(b) C_2, the rotation through π about the Oz-axis of the cylinder;
(c) π_x, the mirror reflection in the plane Oxz; and
(d) π_y, the mirror reflection in the plane Oyz.

These four transformations form the finite group of order four labelled $\mathcal{C}_{2v}$ in the mathematical literature [11]. Four one-dimensional irreducible representations labelled A_1, A_2, B_1 and B_2 are associated with this symmetry group [11] and the S-matrix can be expanded over these irreducible representations. Consequently, each resonance of the circular cylinder is split up into two new distinct resonances. In the literature on scattering by an elliptic cylinder, symmetry considerations are implicitly used as a consequence of the symmetries of the ordinary Mathieu equation. The separation of its solutions into even functions of period π, even functions of period 2π, odd functions of period π and odd functions of period 2π (see [1–3]) corresponds to the separation into functions belonging in the representations A_1, B_1, A_2 and B_2, respectively.

It should be noted that in the context of scattering by simple shapes, the splitting up of resonances has been observed numerically by Moser and Überall [12, 13], but these authors provide neither any analytic description nor any explanation of the phenomenon. In the context of quantum billiards, the splitting up of resonances, linked to the breakdown of a symmetry and to quantum tunnelling, has been noted by several authors (see, for example, [14] and references therein). Splitting up for the elliptic quantum billiard has also been considered long ago [15] and has recently been the subject of new investigations [16, 17] in the context of the semiclassical quantization à la Einstein–Brillouin–Keller (EBK). In these papers, it is also shown that the energy splitting is an exponentially small term which can be recovered by using uniform asymptotic expansions for the solutions of the Mathieu equation.

In the present paper we are mainly concerned with the resonances of an elliptic cylinder. We only consider the Dirichlet boundary condition on the surface of the scatterer. Such a boundary condition corresponds to particle scattering by hard objects in quantum mechanics, ultrasonic wave scattering by soft objects in acoustics and microwave scattering by metallic conductors in electromagnetism. More precisely, we shall derive exponentially improved asymptotic expansions for the poles of the S-matrix for the external problem. Our method is the following.

- We first derive the diffractive part of the Green function of the problem as a sum over the Regge poles of the S-matrix. In order to do that, we use a simplified version of the Sommerfeld–Watson transformation [18, 19] developed by Levy and Keller [9, 20] and Hansen [21].
- We then expand it over the four irreducible representations A_1, A_2, B_1 and B_2 of the symmetry group $\mathcal{C}_{2v}$ of the scatterer. Consequently, resonances of the elliptic cylinder appear as solutions of four transcendental equations involving ordinary Mathieu functions, each one associated with an irreducible representation. Resonances are then naturally classified according to these irreducible representations. It should be noted that the

asymptotic expansions for the resonances are obtained by solving perturbatively the four transcendental equations. The splitting corresponds to an exponentially small term which lies beyond all orders of the asymptotic expansions and can be captured by carefully taking into account Stokes' phenomenon. (For modern aspects of asymptotics beyond all orders and of the Stokes phenomenon, we refer to [22–24].)

- Finally, from the uniform asymptotic expansions of Mathieu functions and of modified Mathieu functions obtained by applying the Langer–Olver method (see [25] and references therein for that method), we obtain asymptotic expansions for Regge poles and exponentially improved asymptotic expansions for resonances.

More precisely, in section 2, we consider the simple example of the circular cylinder in order to present our method, and to gather and improve results scattered in the literature [26, 27]. The diffractive part of the Green function of the external problem is constructed. By using the method of Streifer and Kodis [27], asymptotic expansions for Regge poles are derived from the uniform asymptotic expansions of Hankel functions [2, 28]. Finally, asymptotic expansions for the resonances are obtained. In section 3, we apply our method to the elliptic cylinder. We construct the diffractive part of the Green function, and we establish the transcendental equations that must be satisfied by the Regge poles and by the resonances. We then solve these equations and we derive asymptotic expansions for Regge poles and resonances by using uniform asymptotic expansions for Mathieu and modified Mathieu functions. In the limit of the circular cylinder, we recover the results of section 2. We test our formulae by comparing the results given by the asymptotic expansions with the exact results determined by solving numerically the Helmholtz equation together with the Sommerfeld radiation condition at infinity and the Dirichlet boundary condition on the scatterer. In appendix A, we establish orthonormalization relations for the radial modes arising in the construction of the diffracted Green functions. In appendix B, some symmetry properties of the solutions of the ordinary Mathieu equation are displayed. Finally, in appendices C and D, we derive various asymptotic expansions for Mathieu and modified Mathieu functions.

It should be noted that all the numerical calculations and some tedious algebraic ones have been performed with *Mathematica* [32].

2. The example of the circular cylinder

2.1. Construction of the diffracted Green function

Scattering by an infinite circular cylinder of radius a is considered. The geometry of the problem, as well as the notation used are shown in figure 1. The cylindrical coordinate system (ρ, φ, z) is defined with respect to the symmetry axis of the cylinder.

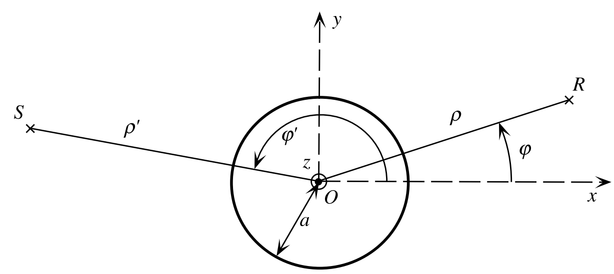

Figure 1. Geometry of the circular cylinder.

The scattering problem is assumed to be independent of the z-coordinate, and thus reduces to a two-dimensional one. Furthermore, an $\exp(-i\omega t)$ time dependence is implicitly assumed. The corresponding Green function $G\left(\rho, \varphi \mid \rho', \varphi'\right)$ is the symmetric solution of the Helmholtz equation

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 G}{\partial\varphi^2} + k^2 G = -\frac{1}{\rho}\delta(\rho-\rho')\delta(\varphi-\varphi') \tag{1}$$

subject to the Dirichlet boundary condition on the scatterer, together with the Sommerfeld radiation condition at infinity. Standard techniques yield a series representation of the solution:

$$G\left(\rho, \varphi \mid \rho', \varphi'\right) = \tfrac{1}{8}\mathrm{i}\sum_{n=0}^{+\infty}\gamma_n\left[H_n^{(2)}(k\rho_<) - \frac{H_n^{(2)}(ka)}{H_n^{(1)}(ka)}H_n^{(1)}(k\rho_<)\right]H_n^{(1)}(k\rho_>)\cos\left[n(\varphi-\varphi')\right]. \tag{2}$$

Here the Neumann factor is defined by $\gamma_0 = 1$ and $\gamma_n = 2$ if $n \neq 0$, $H_n^{(1)}$ and $H_n^{(2)}$ denote, respectively, the Hankel functions of the first and second kinds [2], and $\rho_< = \inf(\rho, \rho')$ while $\rho_> = \sup(\rho, \rho')$. Scattering resonances of the circular cylinder appear as poles of the Green function (or equivalently of the S-matrix), i.e. as the reduced wavenumbers $(ka)_{n,\ell}$ with $\ell = 1, \ldots, +\infty$, solving

$$H_n^{(1)}(ka) = 0 \tag{3}$$

and lying in the fourth quadrant of the complex ka-plane. It should be noted that there is no resonance associated with $n = 0$.

The partial-wave expansion (2) is the exact solution of the problem, but for large ka, it converges very slowly. Furthermore, it does not provide any physical interpretation of the scattering. The well known Sommerfeld–Watson transformation [18, 19] permits one to obtain an alternative representation which displays the various physical (geometrical and diffractive) contributions. More precisely, the expression (2) can be converted into a contour integral in the complex plane ($n \in \mathbb{N} \to \nu \in \mathbb{C}$), with the contour encircling the real positive axis. The deformation of that contour permits one to extract the purely geometrical contribution as well as the diffracted Green function. This one appears as a discrete sum over Regge poles, i.e. the particular values ν_ℓ ($\ell \in \mathbb{N}^*$) of ν, lying in the first quadrant of the complex ν-plane, and satisfying the equation

$$H_\nu^{(1)}(ka) = 0. \tag{4}$$

In the context of scattering by a sphere, Sommerfeld also constructed the diffracted Green function in a more economical way [19] (see [9, 20, 21] for refinements). His method can be applied to scattering by a circular cylinder, and leads us to seek the diffracted Green function G_d in the form

$$G_d(\rho, \varphi) = \sum_{\ell=1}^{+\infty} H_{\nu_\ell}^{(1)}(k\rho)V_\ell(\varphi). \tag{5}$$

Here, in order to simplify our notation, we have suppressed any reference to the coordinates (ρ', φ') of the source point. In fact, the function $V_\ell(\varphi)$ implicitly depends on these coordinates, and G_d is assumed to be symmetrized under the exchange $(\rho, \varphi) \leftrightarrow (\rho', \varphi')$. G_d then automatically satisfies the radiation condition at infinity and the Dirichlet boundary condition on the cylinder. Moreover, it must also satisfy the Helmholtz equation (1), which becomes

$$\sum_{\ell=1}^{+\infty}\left\{\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}H_{\nu_\ell}^{(1)}(k\rho)}{\mathrm{d}\rho}\right) + k^2 H_{\nu_\ell}^{(1)}(k\rho)\right\}V_\ell(\varphi) + \frac{1}{\rho^2}H_{\nu_\ell}^{(1)}(k\rho)V_\ell''(\varphi)$$
$$= -\frac{1}{\rho}\delta(\rho-\rho')\delta(\varphi-\varphi'). \tag{6}$$

It should be noted that the 'eigenfunctions' involved in the expansion (5) do not constitute a complete set of solutions of the boundary value problem under consideration. As a consequence, the discrete sum (5) does not include the geometrical contributions, and so reduces to the exact solution (2) only in the shadow region. Nevertheless, scattering resonances will be deduced from the expression (5), because they are associated with the diffraction phenomenon.

Let us now determine the function $V_\ell(\varphi)$. Since $H^{(1)}_{\nu_\ell}(k\rho)$ is a solution of Bessel's equation

$$\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}U}{\mathrm{d}\rho}\right)+\left(k^2\rho^2-\nu^2\right)U(\rho)=0 \tag{7}$$

with $\nu=\nu_\ell$, equation (6) reduces to

$$\sum_{\ell=1}^{+\infty}\frac{1}{\rho}H^{(1)}_{\nu_\ell}(k\rho)\left[V''_\ell(\varphi)+\nu_\ell^2V_\ell(\varphi)\right]=-\delta(\rho-\rho')\delta(\varphi-\varphi'). \tag{8}$$

We then multiply (8) by $H^{(1)}_{\nu_m}(k\rho)$ (here ν_m is a particular Regge pole), and integrate over the external radial domain $\rho\in[a,+\infty[$ which contains ρ'. The orthonormalization relation (A8) for Hankel functions permits us to write

$$V''_\ell(\varphi)+\nu_\ell^2V_\ell(\varphi)=-\frac{H^{(1)}_{\nu_\ell}(k\rho')}{N_\ell(ka)}\delta(\varphi-\varphi'). \tag{9}$$

The solution of equation (9), defined for $\varphi\in[-\pi,\pi]$, can be sought in the form

$$V_\ell(\varphi)=\Theta(\varphi'-\varphi)\left[A\cos\nu_\ell\varphi+B\sin\nu_\ell\varphi\right]+\Theta(\varphi-\varphi')\left[C\cos\nu_\ell\varphi+D\sin\nu_\ell\varphi\right] \tag{10}$$

where Θ denotes the Heavyside step function, and A, B, C and D depend on (ρ',φ'). By replacing (10) into (9), and then by identifying the terms involving, respectively, $\delta'(\varphi-\varphi')$ and $\delta(\varphi-\varphi')$, we obtain the following pair of equations:

$$(A-C)\cos\nu_\ell\varphi'+(B-D)\sin\nu_\ell\varphi'=0 \tag{11}$$

which expresses the continuity of $V_\ell(\varphi)$ at $\varphi=\varphi'$, and

$$(B-D)\cos\nu_\ell\varphi'-(A-C)\sin\nu_\ell\varphi'=\frac{H^{(1)}_{\nu_\ell}(k\rho')}{\nu_\ell N_\ell(ka)} \tag{12}$$

which corresponds to the fact that $V'_\ell(\varphi)$ has a jump at $\varphi=\varphi'$, equal to the right-hand side of (12). Furthermore, by using the condition that $V_\ell(\varphi)$ and $V'_\ell(\varphi)$ must be single-valued functions (they take the same values at $\varphi=-\pi$ and $\varphi=\pi$), we determine the constants A, B, C and D, and we obtain

$$V_\ell(\varphi)=-\frac{H^{(1)}_{\nu_\ell}(k\rho')}{2\nu_\ell N_\ell(ka)\sin\nu_\ell\pi}\big[\Theta(\varphi'-\varphi)\cos\nu_\ell(\pi-\varphi'+\varphi)$$
$$+\Theta(\varphi-\varphi')\cos\nu_\ell(\pi+\varphi'-\varphi)\big]. \tag{13}$$

Finally, by replacing (13) into (5), and by taking into account the expression (A12) of the normalization factor $N_\ell(ka)$, we obtain for the diffracted Green function

$$G_d(\rho,\varphi\mid\rho',\varphi')=\tfrac{1}{4}\mathrm{i}\pi\sum_{\ell=1}^{+\infty}\frac{H^{(2)}_{\nu_\ell}(ka)H^{(1)}_{\nu_\ell}(k\rho')H^{(1)}_{\nu_\ell}(k\rho)\cos\nu_\ell(\varphi'-\varphi\pm\pi)}{\left[\partial H^{(1)}_\nu(ka)/\partial\nu\right]_{\nu=\nu_\ell}\sin\nu_\ell\pi} \tag{14}$$

where the upper (respectively, the lower) sign applies when $\varphi>\varphi'$ (respectively, $\varphi<\varphi'$). Here, the reference to the coordinates (ρ',φ') of the source point as well as the symmetry of G_d under the exchange $(\rho,\varphi)\leftrightarrow(\rho',\varphi')$ appear explicitly.

2.2. Asymptotic expansions for Regge poles

Now, we shall determine asymptotic expansions (for $ka \to \infty$) for the Regge poles $\nu_\ell(ka)$, by solving equation (4). With this aim in mind, we need the uniform asymptotic expansion for the Hankel function of the first kind [2, 28] obtained by using the Langer–Olver method [25] and which is valid for $|\nu| \to \infty$,

$$H_\nu^{(1)}(k\rho) = 2\sqrt{2}\left(\frac{\nu^2\zeta}{\nu^2 - k^2\rho^2}\right)^{1/4}\left\{\frac{\mathrm{e}^{-\mathrm{i}\pi/3}\mathrm{Ai}\left(\mathrm{e}^{2\mathrm{i}\pi/3}\nu^{2/3}\zeta\right)}{\nu^{1/3}}\sum_{s=0}^{\infty}\frac{A_s(\zeta)}{\nu^{2s}} + \frac{\mathrm{e}^{\mathrm{i}\pi/3}\mathrm{Ai}'\left(\mathrm{e}^{2\mathrm{i}\pi/3}\nu^{2/3}\zeta\right)}{\nu^{5/3}}\sum_{s=0}^{\infty}\frac{B_s(\zeta)}{\nu^{2s}}\right\}. \tag{15}$$

Here, ζ is given by

$$\tfrac{2}{3}\zeta^{3/2} = -\int_1^{k\rho/\nu}\left(\frac{1-z^2}{z^2}\right)^{1/2}\mathrm{d}z = \ln\left(\frac{\nu + \left(\nu^2 - k^2\rho^2\right)^{1/2}}{k\rho}\right) - \left(\frac{\nu^2 - k^2\rho^2}{\nu^2}\right)^{1/2} \tag{16}$$

and is chosen so that it is real when ν is real and positive, and $k\rho/\nu \in [0, 1]$. Furthermore, the coefficients $A_s(\zeta)$ and $B_s(\zeta)$ are defined by recurrence relations and are given explicitly in [2, 28]. In particular, we have

$$A_0(\zeta) = 1 \tag{17a}$$

$$B_0(\zeta) = -\frac{5}{48\zeta^2} + \frac{1}{24\zeta^{1/2}}\left[\frac{5\nu^3}{\left(\nu^2 - k^2\rho^2\right)^{3/2}} - \frac{3\nu}{\left(\nu^2 - k^2\rho^2\right)^{1/2}}\right]. \tag{17b}$$

Let us first consider the leading term of (15) where we take $\rho = a$. Thus we have

$$H_\nu^{(1)}(ka) = 2\sqrt{2}\left(\frac{\nu^2\zeta}{\nu^2 - k^2a^2}\right)^{1/4}\frac{\mathrm{e}^{-\mathrm{i}\pi/3}\mathrm{Ai}\left(\mathrm{e}^{2\mathrm{i}\pi/3}\nu^{2/3}\zeta\right)}{\nu^{1/3}}\left[1 + \underset{|\nu|\to\infty}{\mathrm{O}}\left(\frac{1}{\nu}\right)\right]. \tag{18}$$

In the context of this approximation, equation (4) reduces to $\mathrm{Ai}\left(\mathrm{e}^{2\mathrm{i}\pi/3}\nu^{2/3}\zeta\right) \approx 0$. By introducing the real negative zeros x_ℓ ($\ell \in \mathbb{N}^*$) of the Airy function $\mathrm{Ai}(x)$ [2], we see that the Regge poles satisfy

$$\nu\zeta^{3/2} \approx -x_\ell^{3/2}. \tag{19}$$

Equation (19) implies that $\frac{2}{3}\zeta^{3/2} = \mathrm{O}_{|\nu|\to\infty}(1/\nu)$. Thus, it follows from (16) that $\nu_\ell \approx ka$. Now, equation (19) can be solved perturbatively, and we find, to this degree of approximation,

$$\begin{aligned}\nu_\ell(ka) \approx \nu_\ell^{\mathrm{approx}}(ka) = ka &- \mathrm{e}^{\mathrm{i}\pi/3}x_\ell\left(\tfrac{1}{2}ka\right)^{1/3} + \frac{\mathrm{e}^{2\mathrm{i}\pi/3}x_\ell^2}{60}\left(\tfrac{1}{2}ka\right)^{-1/3} - \frac{x_\ell^3}{1400}\left(\tfrac{1}{2}ka\right)^{-1} \\ &- \frac{281\mathrm{e}^{\mathrm{i}\pi/3}x_\ell^4}{4536\,000}\left(\tfrac{1}{2}ka\right)^{-5/3} - \frac{73\,769\mathrm{e}^{2\mathrm{i}\pi/3}x_\ell^5}{10\,478\,160\,000}\left(\tfrac{1}{2}ka\right)^{-7/3} \\ &+ \frac{93\,617x_\ell^6}{100\,900\,800\,000}\left(\tfrac{1}{2}ka\right)^{-3} + \underset{ka\to\infty}{\mathrm{O}}\left[(ka)^{-11/3}\right].\end{aligned} \tag{20}$$

Now, we consider the two first leading terms of (15):

$$H_\nu^{(1)}(ka) = 2\sqrt{2}\left(\frac{\nu^2\zeta}{\nu^2 - k^2a^2}\right)^{1/4}\left\{\frac{\mathrm{e}^{-\mathrm{i}\pi/3}\mathrm{Ai}\left(\mathrm{e}^{2\mathrm{i}\pi/3}\nu^{2/3}\zeta\right)}{\nu^{1/3}} + \frac{\mathrm{e}^{\mathrm{i}\pi/3}\mathrm{Ai}'\left(\mathrm{e}^{2\mathrm{i}\pi/3}\nu^{2/3}\zeta\right)}{\nu^{5/3}}B_0(\zeta)\right\}\left[1 + \underset{|\nu|\to\infty}{\mathrm{O}}\left(\frac{1}{\nu^2}\right)\right]. \tag{21}$$

To this new degree of approximation, the condition $H_\nu^{(1)}(ka) = 0$ can be written as

$$\frac{\mathrm{Ai}\left[\mathrm{e}^{2\mathrm{i}\pi/3}\nu^{2/3}\zeta\right]}{\mathrm{Ai}'\left[\mathrm{e}^{2\mathrm{i}\pi/3}\nu^{2/3}\zeta\right]} \approx -\mathrm{e}^{2\mathrm{i}\pi/3}\nu^{-4/3}B_0(\zeta). \tag{22}$$

Regge poles can then be sought as the solutions of

$$\nu\zeta^{3/2} \approx -(x_\ell + \delta x)^{3/2} \tag{23}$$

where δx can be obtained from (22). By expanding in (22) $\mathrm{Ai}(x)$ and $\mathrm{Ai}'(x)$ in Taylor series about x_ℓ, we obtain

$$\delta x = -\mathrm{e}^{2\mathrm{i}\pi/3}\nu^{-4/3}B_0(\zeta). \tag{24}$$

Equation (23) can be solved step by step, and we finally obtain, for Regge poles, the following asymptotic expansions:

$$\begin{aligned}\nu_\ell(ka) = ka &- \mathrm{e}^{\mathrm{i}\pi/3}x_\ell\left(\tfrac{1}{2}ka\right)^{1/3} + \tfrac{1}{60}\mathrm{e}^{2\mathrm{i}\pi/3}x_\ell^2\left(\tfrac{1}{2}ka\right)^{-1/3} - \frac{x_\ell^3+10}{1400}\left(\tfrac{1}{2}ka\right)^{-1} \\ &- \frac{\mathrm{e}^{\mathrm{i}\pi/3}\left(281x_\ell^4 + 10\,440x_\ell\right)}{4536\,000}\left(\tfrac{1}{2}ka\right)^{-5/3} \\ &- \frac{\mathrm{e}^{2\mathrm{i}\pi/3}\left(73\,769x_\ell^5 + 6624\,900x_\ell^2\right)}{10\,478\,160\,000}\left(\tfrac{1}{2}ka\right)^{-7/3} \\ &+ \frac{93\,617x_\ell^6 + 16\,495\,400x_\ell^3 - 1744\,600}{100\,900\,800\,000}\left(\tfrac{1}{2}ka\right)^{-3} + \underset{ka\to+\infty}{\mathrm{O}}\left[(ka)^{-11/3}\right].\end{aligned} \tag{25}$$

Up to this order of the asymptotic expansion (25), the terms corresponding to $s = 0$ into (15) are sufficient. Moreover, in comparison with the expansion for Regge poles obtained by Streifer and Kodis [27], it should be noted that (25) includes two additional terms.

2.3. Asymptotic expansions for resonances

It appears from the expression for the diffracted Green function (14) that resonances are obtained by solving $\sin[\pi\nu_\ell(ka)] = 0$. The condition for resonance can then be expressed as

$$\nu_\ell(ka) = n \tag{26}$$

with $n \in \mathbb{N}^*$. By taking for $\nu_\ell(ka)$ its asymptotic expansion (25) and then by inverting equation (26), we obtain

$$\begin{aligned}(ka)_{n,\ell} = n &+ \mathrm{e}^{\mathrm{i}\pi/3}x_\ell\left(\tfrac{1}{2}n\right)^{1/3} + \frac{3\mathrm{e}^{2\mathrm{i}\pi/3}x_\ell^2}{20}\left(\tfrac{1}{2}n\right)^{-1/3} + \frac{x_\ell^3+10}{1400}\left(\tfrac{1}{2}n\right)^{-1} \\ &+ \frac{\mathrm{e}^{\mathrm{i}\pi/3}\left(479x_\ell^4 - 40x_\ell\right)}{504\,000}\left(\tfrac{1}{2}n\right)^{-5/3} - \frac{\mathrm{e}^{2\mathrm{i}\pi/3}\left(20\,231x_\ell^5 + 55\,100x_\ell^2\right)}{129\,360\,000}\left(\tfrac{1}{2}n\right)^{-7/3} \\ &+ \underset{n\to+\infty}{\mathrm{O}}\left(n^{-3}\right).\end{aligned} \tag{27}$$

The previous expression can be recovered from the uniform asymptotic expansions for the zeros of the Hankel function $H_\nu^{(1)}(z)$ given by Olver [28].

3. Diffraction by an elliptic cylinder

3.1. Geometry of the scatterer, symmetry considerations and the exact Green function

Let us now consider scattering by an infinite cylinder of elliptic cross section. This problem is assumed to be independent of the z-coordinate along the axis Oz of the cylinder, and thus reduces to a two-dimensional one. The geometry of the scatterer is then well described by the elliptic coordinates (ξ, η) related to the rectangular coordinates (x, y) by the transformation (see figure 2)

$$x = c\cosh\xi\,\cos\eta \qquad y = c\sinh\xi\,\sin\eta. \tag{28}$$

where $0 \leqslant \xi < \infty$ and $-\pi \leqslant \eta \leqslant \pi$. The equation $\xi = \xi_0$ defines the surface of an elliptic cylinder whose eccentricity is $1/\cosh\xi_0$. The limiting cases $\xi_0 = 0$ and $\xi_0 \to +\infty$ correspond, respectively, to the strip and to the circular cylinder [2, 3].

It should be noted that the transition from the circular cylinder to the elliptic one corresponds to the breaking of the $O(2)$-symmetry (invariance under any rotation about the Oz-axis). However (see figure 3), the elliptic cylinder remains invariant under four symmetry transformations: E, the identity transformation ($\eta \to \eta$); C_2, the rotation through π about the

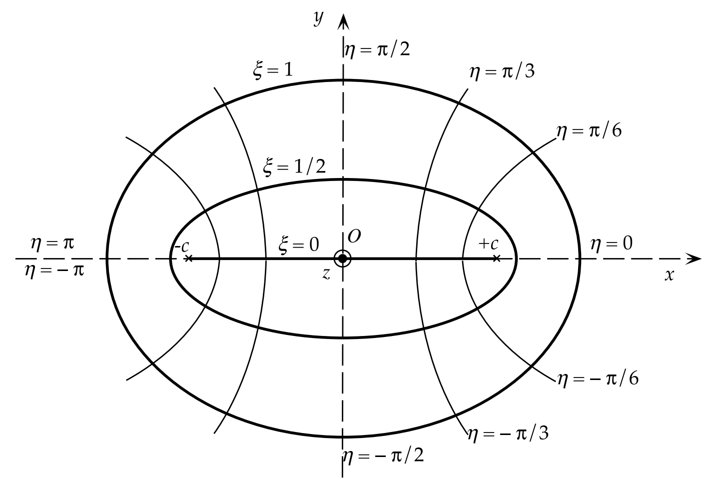

Figure 2. Elliptic coordinates.

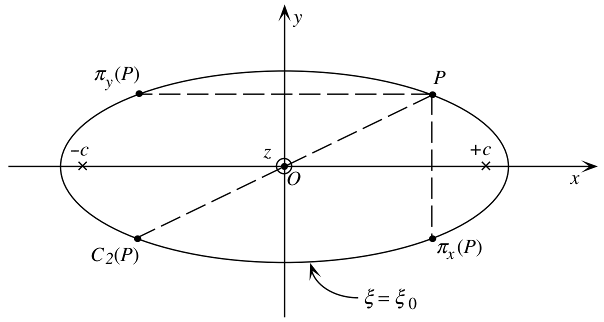

Figure 3. Geometry of the elliptic cylinder.

Table 1. Character table of $\mathcal{C}_{2v}$.

$\mathcal{C}_{2v}$:	E	C_2	π_x	π_y
A_1	1	1	1	1
A_2	1	1	−1	−1
B_1	1	−1	1	−1
B_2	1	−1	−1	1

Oz-axis ($\eta \to \pi + \eta$); π_x, the mirror reflection in the plane Oxz ($\eta \to -\eta$); π_y, the mirror reflection in the plane Oyz ($\eta \to \pi - \eta$). These four transformations form the finite group $\mathcal{C}_{2v}$, which is the symmetry group of the scatterer.

Four one-dimensional irreducible representations labelled A_1, A_2, B_1 and B_2 are associated with the symmetry group $\mathcal{C}_{2v}$ [11]. In the representation A_1 (respectively, A_2, B_1 and B_2), the group elements E, C_2, π_x and π_y are represented by 1×1 matrices given in the A_1 (respectively, A_2, B_1 and B_2) row of the character table (table 1). Consequently, any function V of the 'angular' coordinate η can be expanded over these four irreducible representations as

$$V(\eta) = V^{(A_1)}(\eta) + V^{(A_2)}(\eta) + V^{(B_1)}(\eta) + V^{(B_2)}(\eta) \tag{29}$$

with

$$EV^{(A_1)} = V^{(A_1)} \qquad C_2V^{(A_1)} = V^{(A_1)} \qquad \pi_x V^{(A_1)} = V^{(A_1)} \qquad \pi_y V^{(A_1)} = V^{(A_1)} \tag{30a}$$

$$EV^{(A_2)} = V^{(A_2)} \qquad C_2V^{(A_2)} = V^{(A_2)} \qquad \pi_x V^{(A_2)} = -V^{(A_2)} \qquad \pi_y V^{(A_2)} = -V^{(A_2)} \tag{30b}$$

$$EV^{(B_1)} = V^{(B_1)} \qquad C_2V^{(B_1)} = -V^{(B_1)} \qquad \pi_x V^{(B_1)} = V^{(B_1)} \qquad \pi_y V^{(B_1)} = -V^{(B_1)} \tag{30c}$$

$$EV^{(B_2)} = V^{(B_2)} \qquad C_2V^{(B_2)} = -V^{(B_2)} \qquad \pi_x V^{(B_2)} = -V^{(B_2)} \qquad \pi_y V^{(B_2)} = V^{(B_2)}. \tag{30d}$$

Furthermore, the components $V^{(A_1)}$, $V^{(A_2)}$, $V^{(B_1)}$ and $V^{(B_2)}$ satisfying (30) are given explicitly by

$$V^{(A_1)}(\eta) = \tfrac{1}{4}\left(E + C_2 + \pi_x + \pi_y\right) V(\eta) \tag{31a}$$

$$V^{(A_2)}(\eta) = \tfrac{1}{4}\left(E + C_2 - \pi_x - \pi_y\right) V(\eta) \tag{31b}$$

$$V^{(B_1)}(\eta) = \tfrac{1}{4}\left(E - C_2 + \pi_x - \pi_y\right) V(\eta) \tag{31c}$$

$$V^{(B_2)}(\eta) = \tfrac{1}{4}\left(E - C_2 - \pi_x + \pi_y\right) V(\eta). \tag{31d}$$

The Green function $G(\xi, \eta \mid \xi', \eta')$ associated with the scattering problem is the symmetric solution of the Helmholtz equation in elliptic coordinates,

$$\frac{\partial^2 G}{\partial \xi^2} + \frac{\partial^2 G}{\partial \eta^2} + (kc)^2(\cosh^2\xi - \cos^2\eta)G = -\delta(\xi - \xi')\delta(\eta - \eta') \tag{32}$$

still subject to the Dirichlet boundary condition on the scatterer together with the Sommerfeld radiation condition at infinity. It can be expanded over the representations A_1, A_2, B_1 and B_2 as

$$G(\xi, \eta \mid \xi', \eta') = G^{(A_1)}(\xi, \eta \mid \xi', \eta') + G^{(A_2)}(\xi, \eta \mid \xi', \eta') + G^{(B_1)}(\xi, \eta \mid \xi', \eta') + G^{(B_2)}(\xi, \eta \mid \xi', \eta') \tag{33}$$

with

$$G^{(A_1)}(\xi,\eta\mid\xi',\eta') = \tfrac{1}{4}\mathrm{i}\sum_{r=0}^{+\infty}\left[Mc_{2r}^{(4)}(\xi_<,\theta) - \frac{Mc_{2r}^{(4)}(\xi_0,\theta)}{Mc_{2r}^{(3)}(\xi_0,\theta)}Mc_{2r}^{(3)}(\xi_<,\theta)\right]Mc_{2r}^{(3)}(\xi_>,\theta) \times ce_{2r}(\eta,\theta)\,ce_{2r}(\eta',\theta) \qquad (34a)$$

$$G^{(A_2)}(\xi,\eta\mid\xi',\eta') = \tfrac{1}{4}\mathrm{i}\sum_{r=1}^{+\infty}\left[Ms_{2r}^{(4)}(\xi_<,\theta) - \frac{Ms_{2r}^{(4)}(\xi_0,\theta)}{Ms_{2r}^{(3)}(\xi_0,\theta)}Ms_{2r}^{(3)}(\xi_<,\theta)\right]Ms_{2r}^{(3)}(\xi_>,\theta) \times se_{2r}(\eta,\theta)\,se_{2r}(\eta',\theta) \qquad (34b)$$

$$G^{(B_1)}(\xi,\eta\mid\xi',\eta') = \tfrac{1}{4}\mathrm{i}\sum_{r=0}^{+\infty}\left[Mc_{2r+1}^{(4)}(\xi_<,\theta) - \frac{Mc_{2r+1}^{(4)}(\xi_0,\theta)}{Mc_{2r+1}^{(3)}(\xi_0,\theta)}Mc_{2r+1}^{(3)}(\xi_<,\theta)\right]Mc_{2r+1}^{(3)}(\xi_>,\theta) \times ce_{2r+1}(\eta,\theta)\,ce_{2r+1}(\eta',\theta) \qquad (34c)$$

$$G^{(B_2)}(\xi,\eta\mid\xi',\eta') = \tfrac{1}{4}\mathrm{i}\sum_{r=0}^{+\infty}\left[Ms_{2r+1}^{(4)}(\xi_<,\theta) - \frac{Ms_{2r+1}^{(4)}(\xi_0,\theta)}{Ms_{2r+1}^{(3)}(\xi_0,\theta)}Ms_{2r+1}^{(3)}(\xi_<,\theta)\right]Ms_{2r+1}^{(3)}(\xi_>,\theta) \times se_{2r+1}(\eta,\theta)\,se_{2r+1}(\eta',\theta). \qquad (34d)$$

Here, $\xi_< = \inf(\xi,\xi')$ and $\xi_> = \sup(\xi,\xi')$, while $\theta = (kc/2)^2$. The mode solutions $Mc_n^{(3)}(\xi,\theta)\,ce_n(\eta,\theta)$, $Ms_n^{(3)}(\xi,\theta)\,se_n(\eta,\theta)$, $Mc_n^{(4)}(\xi,\theta)\,ce_n(\eta,\theta)$ and $Ms_n^{(4)}(\xi,\theta)\,se_n(\eta,\theta)$ appearing in expressions (34) are defined in [2]. They arise when one seeks the solutions of the Helmholtz equation $(\Delta + k^2)\Psi(\xi,\eta) = 0$ by separation of variables, i.e. in the form $\Psi(\xi,\eta) = U(\xi)V(\eta)$. In this case, $U(\xi)$ and $V(\eta)$, respectively, satisfy the modified Mathieu equation

$$U''(\xi) - (kc)^2\left(b^2 - \cosh^2\xi\right)U(\xi) = 0 \qquad (35)$$

and the ordinary Mathieu equation

$$V''(\eta) + (kc)^2\left(b^2 - \cos^2\eta\right)V(\eta) = 0. \qquad (36)$$

For a given kc, there exists a countably infinite set $\left(b_r^{(A_1)}\right)_{r\in\mathbb{N}}$ of characteristic values of the separation constant b which yields solutions of (36) belonging in the irreducible representation A_1 of $\mathcal{C}_{2v}$. More precisely, for a given $r \in \mathbb{N}$, the characteristic value $b_r^{(A_1)}$ is associated with the Mathieu function ce_{2r} which forms a basis for the one-dimensional representation A_1. Any solution of equation (35), where we take $b = b_r^{(A_1)}$, can then be written as a linear combination of the two modified Mathieu functions $Mc_{2r}^{(3)}$ (outgoing solution) and $Mc_{2r}^{(4)}$ (incoming solution). Similarly, there exist three other countably infinite sets $\left(b_r^{(A_2)}\right)_{r\in\mathbb{N}^*}$, $\left(b_r^{(B_1)}\right)_{r\in\mathbb{N}}$ and $\left(b_r^{(B_2)}\right)_{r\in\mathbb{N}}$ of characteristic values of the separation constant b associated with solutions of (36) belonging, respectively, in the irreducible representations A_2, B_1 and B_2. The corresponding Mathieu functions and modified Mathieu functions are given in table 2.

Furthermore, it should be noted that our notation for characteristic values are related to the usual ones [2] by

$$b_r^{(A_1)}(kc) = \left(\frac{a_{2r}((kc/2)^2)}{(kc)^2} + \frac{1}{2}\right)^{1/2} \qquad b_r^{(B_1)}(kc) = \left(\frac{a_{2r+1}((kc/2)^2)}{(kc)^2} + \frac{1}{2}\right)^{1/2}$$
$$b_r^{(A_2)}(kc) = \left(\frac{b_{2r}((kc/2)^2)}{(kc)^2} + \frac{1}{2}\right)^{1/2} \qquad b_r^{(B_2)}(kc) = \left(\frac{b_{2r+1}((kc/2)^2)}{(kc)^2} + \frac{1}{2}\right)^{1/2}. \qquad (37)$$

Table 2. Irreducible representations of $\mathcal{C}_{2v}$ and corresponding Mathieu functions.

	Characteristic value	Ordinary Mathieu function	Outgoing modified Mathieu function	Incoming modified Mathieu function
A_1	$b_r^{(A_1)}(kc)$	$ce_{2r}\left(\eta, (kc/2)^2\right)$	$Mc_{2r}^{(3)}\left(\xi, (kc/2)^2\right)$	$Mc_{2r}^{(4)}\left(\xi, (kc/2)^2\right)$
A_2	$b_r^{(A_2)}(kc)$	$se_{2r}\left(\eta, (kc/2)^2\right)$	$Ms_{2r}^{(3)}\left(\xi, (kc/2)^2\right)$	$Ms_{2r}^{(4)}\left(\xi, (kc/2)^2\right)$
B_1	$b_r^{(B_1)}(kc)$	$ce_{2r+1}\left(\eta, (kc/2)^2\right)$	$Mc_{2r+1}^{(3)}\left(\xi, (kc/2)^2\right)$	$Mc_{2r+1}^{(4)}\left(\xi, (kc/2)^2\right)$
B_2	$b_r^{(B_2)}(kc)$	$se_{2r+1}\left(\eta, (kc/2)^2\right)$	$Ms_{2r+1}^{(3)}\left(\xi, (kc/2)^2\right)$	$Ms_{2r+1}^{(4)}\left(\xi, (kc/2)^2\right)$

Scattering resonances of the elliptic cylinder appear as the poles of the Green function (33) (or equivalently of the S-matrix), i.e. as the reduced wavenumbers kc solving

$$Mc_{2r}^{(3)}(\xi_0, (kc/2)^2) = 0 \qquad r \in \mathbb{N} \qquad (A_1) \tag{38a}$$

$$Ms_{2r}^{(3)}(\xi_0, (kc/2)^2) = 0 \qquad r \in \mathbb{N}^* \qquad (A_2) \tag{38b}$$

$$Mc_{2r+1}^{(3)}(\xi_0, (kc/2)^2) = 0 \qquad r \in \mathbb{N} \qquad (B_1) \tag{38c}$$

$$Ms_{2r+1}^{(3)}(\xi_0, (kc/2)^2) = 0 \qquad r \in \mathbb{N} \qquad (B_2) \tag{38d}$$

and lying in the fourth quadrant of the complex kc-plane. It should be noted that in the representation A_1, there is no resonance associated with $r = 0$.

The Green function (2) corresponding to scattering by a circular cylinder of radius a can be recovered from the Green function (33). This can be done by taking both the limits $\xi_0 \to +\infty$ and $kc \to 0$, while keeping $(kc/2)\exp\xi_0$ constant and equal to the reduced wavenumber ka. Indeed:

(a) For large values of ξ, $\rho = (x^2 + y^2)^{1/2}$ is approximately $(c/2)\exp\xi$ (see (28)) and the modified Mathieu equation (35) reduces to the Bessel equation (7), with

$$\nu = kc\left(b^2 - \tfrac{1}{2}\right)^{1/2}. \tag{39}$$

(b) Then, for $kc \to 0$, we can make the substitution $\eta \to \varphi$ in the Mathieu equation (36), which reduces to $V''(\varphi) + \nu^2 V(\varphi) = 0$, and which admits periodic solutions only for $\nu = n \in \mathbb{N}$.

From these remarks and the definitions of the Mathieu and modified Mathieu functions [2], we obtain the following correspondences:

$$Mc_n^{(3)}(\xi, (kc/2)^2) \sim Ms_n^{(3)}(\xi, (kc/2)^2) \longrightarrow H_n^{(1)}(k\rho) \tag{40a}$$

$$Mc_n^{(4)}(\xi, (kc/2)^2) \sim Ms_n^{(4)}(\xi, (kc/2)^2) \longrightarrow H_n^{(2)}(k\rho) \tag{40b}$$

$$ce_n(\eta, (kc/2)^2) \longrightarrow \sqrt{\tfrac{1}{2}\gamma_n}\cos n\varphi \tag{40c}$$

$$se_n(\eta, (kc/2)^2) \longrightarrow \sin n\varphi. \tag{40d}$$

Thus, by substituting equations (40) in the Green function (33), we recover (2).

The four equations (38a)–(38d) provide an algebraic classification of resonances. They can be solved numerically by using a shooting method. Since the modified Mathieu functions are not built in *Mathematica*, this is a time-consuming task. Indeed, in order to scan suitable regions of the complex kc-plane, we need to construct, with great accuracy and for a great number of kc, outgoing solutions of the modified Mathieu equation (35). In figure 4 we present

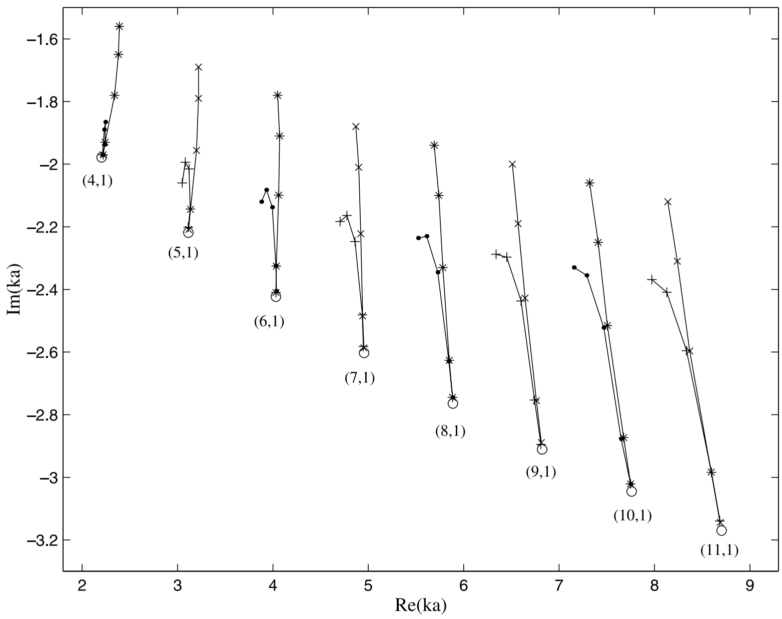

Figure 4. Resonances in the complex ka-plane. ○, circular cylinder; $*$, A_1; •, A_2; ×, B_1; +, B_2.

the locations of resonances in the complex ka-plane, where the reduced wavenumber ka is defined by $ka = (kc/2)\exp\xi_0$. Scattering resonances are plotted for the following values of ξ_0: $\xi_0 \to \infty$, $\xi_0 = 1$, $\xi_0 = 0.5$, $\xi_0 = 0.2$, $\xi_0 = 0.1$ and $\xi_0 = 0.05$. The splitting up of resonances linked to the breaking of the $O(2)$-symmetry is thus displayed.

3.2. *Construction of the diffracted Green function*

As for the case of the circular cylinder in subsection 2.1, we shall provide the expression for the diffracted Green function associated with scattering by an elliptic cylinder. We first extend the approach of Levy [9] and then expand the diffracted Green function over the representations of $\mathcal{C}_{2v}$.

Regge poles of the elliptic cylinder are the particular values b_ℓ ($\ell \in \mathbb{N}^*$) of the separation constant b, lying in the first quadrant of the complex b-plane, and solving the equation

$$U^{(1)}(\xi_0, b) = 0 \tag{41}$$

where $U^{(1)}(\xi, b)$ is an outgoing solution of equation (35). The diffracted Green function is then sought as a sum over the Regge poles b_ℓ, in the form

$$G_d(\xi, \eta) = \sum_{\ell=1}^{+\infty} U_\ell^{(1)}(\xi) V_\ell(\eta). \tag{42}$$

Here, in order to simplify our notation, any reference to the coordinates (ξ', η') of the source point has been omitted, the dependence on kc is implicitly assumed, while the dependence on the Regge pole b_ℓ is expressed by using the index ℓ. Because G_d is assumed to be symmetrized under the exchange $(\xi, \eta) \leftrightarrow (\xi', \eta')$, it automatically satisfies the boundary conditions at infinity and on the cylinder. Now, by substituting (42) into (32), and by using the orthonormalization relation (A13) for the functions $U_\ell^{(1)}(\xi)$, we find that $V_\ell(\eta)$ satisfies

$$V_\ell''(\eta) + (kc)^2(b_\ell^2 - \cos^2\eta)V_\ell(\eta) = -\frac{U_\ell^{(1)}(\xi')}{N_\ell(\xi_0)}\delta(\eta - \eta') \tag{43}$$

where $N_\ell(\xi_0)$ is defined by (A18). Let us now consider the two linearly independent solutions $c(\eta, b_\ell) = c_\ell(\eta)$ and $s(\eta, b_\ell) = s_\ell(\eta)$ of equation (36), normalized in such a way that (see also (B1))

$$\begin{aligned} c_\ell(0) &= 1 \qquad c_\ell'(0) = 0 \\ s_\ell(0) &= 0 \qquad s_\ell'(0) = 1. \end{aligned} \tag{44}$$

We then assume that any solution of (43) can be written in the form

$$V_\ell(\eta) = \Theta(\eta' - \eta)\,[Ac_\ell(\eta) + Bs_\ell(\eta)] + \Theta(\eta - \eta')\,[Cc_\ell(\eta) + Ds_\ell(\eta)]. \tag{45}$$

By substituting it into (43), and by using the single valuedness of $V_\ell(\eta)$ and $V_\ell'(\eta)$, we identify the constants A, B, C and D and we obtain

$$\begin{aligned} V_\ell(\eta) = \frac{U_\ell^{(1)}(\xi')}{2N_\ell(\xi_0)}\Bigg\{ & \frac{s_\ell'(\pi)}{c_\ell'(\pi)}\,c_\ell(\eta')\,c_\ell(\eta) - \frac{c_\ell(\pi)}{s_\ell(\pi)}s_\ell(\eta')\,s_\ell(\eta) \\ & +\Theta(\eta - \eta')\left[s_\ell(\eta')\,c_\ell(\eta) - c_\ell(\eta')\,s_\ell(\eta)\right] \\ & -\Theta(\eta' - \eta)\left[s_\ell(\eta')\,c_\ell(\eta) - c_\ell(\eta')\,s_\ell(\eta)\right]\Bigg\}. \end{aligned} \tag{46}$$

By using the relations (B4) and (B5), we finally obtain the diffracted Green function as

$$G_d(\xi, \eta \mid \xi', \eta') = \sum_{\ell=1}^{+\infty} \frac{U_\ell^{(1)}(\xi')U_\ell^{(1)}(\xi)}{4N_\ell(\xi_0)}\left[\frac{c_\ell(\eta' \pm \pi)\,c_\ell(\eta)}{c_\ell(\pi/2)\,c_\ell'(\pi/2)} - \frac{s_\ell(\eta' \pm \pi)\,s_\ell(\eta)}{s_\ell(\pi/2)\,s_\ell'(\pi/2)}\right] \tag{47}$$

where the upper (respectively, the lower) signs apply when $\eta > \eta'$ (respectively, $\eta < \eta'$).

Now, we wish to expand the diffracted Green function (47) over the four irreducible representations of the symmetry group $\mathcal{C}_{2v}$. To carry out this calculation, we consider $\eta' \in [0, \pi/2]$. From a physical point of view, this restriction does not involve any loss of generality. Moreover, we limit our study to $\eta \in [0, \pi/2]$: the behaviour of the functions $G_d^{(A_1)}$, $G_d^{(A_2)}$, $G_d^{(B_1)}$ and $G_d^{(B_2)}$ over the whole range $\eta \in [-\pi, \pi]$ can easily be deduced from their symmetry properties (30). From definitions (29) and (31), and by using relations (B4) and (B5), we obtain

$$\begin{aligned} G_d(\xi, \eta \mid \xi', \eta') = {} & G_d^{(A_1)}(\xi, \eta \mid \xi', \eta') + G_d^{(A_2)}(\xi, \eta \mid \xi', \eta') + G_d^{(B_1)}(\xi, \eta \mid \xi', \eta') \\ & +G_d^{(B_2)}(\xi, \eta \mid \xi', \eta') \end{aligned} \tag{48}$$

with

$$G_d^{(A_1)}(\xi, \eta \mid \xi', \eta') = \sum_{\ell=1}^{+\infty} \frac{U_\ell^{(1)}(\xi')U_\ell^{(1)}(\xi)}{4N_\ell(\xi_0)}\left[\frac{s_\ell'(\pi/2)}{c_\ell'(\pi/2)}c_\ell(\eta_>) - s_\ell(\eta_>)\right]c_\ell(\eta_<) \tag{49a}$$

$$G_d^{(A_2)}(\xi, \eta \mid \xi', \eta') = -\sum_{\ell=1}^{+\infty} \frac{U_\ell^{(1)}(\xi')U_\ell^{(1)}(\xi)}{4N_\ell(\xi_0)}\left[\frac{c_\ell(\pi/2)}{s_\ell(\pi/2)}s_\ell(\eta_>) - c_\ell(\eta_>)\right]s_\ell(\eta_<) \tag{49b}$$

$$G_d^{(B_1)}(\xi, \eta \mid \xi', \eta') = \sum_{\ell=1}^{+\infty} \frac{U_\ell^{(1)}(\xi')U_\ell^{(1)}(\xi)}{4N_\ell(\xi_0)}\left[\frac{s_\ell(\pi/2)}{c_\ell(\pi/2)}c_\ell(\eta_>) - s_\ell(\eta_>)\right]c_\ell(\eta_<) \tag{49c}$$

$$G_d^{(B_2)}(\xi, \eta \mid \xi', \eta') = -\sum_{\ell=1}^{+\infty} \frac{U_\ell^{(1)}(\xi')U_\ell^{(1)}(\xi)}{4N_\ell(\xi_0)}\left[\frac{c_\ell'(\pi/2)}{s_\ell'(\pi/2)}s_\ell(\eta_>) - c_\ell(\eta_>)\right]s_\ell(\eta_<). \tag{49d}$$

Here, $\eta_< = \inf(\eta, \eta')$, while $\eta_> = \sup(\eta, \eta')$. The previous expressions clearly display the conditions for resonance associated with each irreducible representation of the symmetry group $\mathcal{C}_{2v}$. More precisely, the transcendental equation $c'_\ell(\pi/2) = 0$ (respectively, $s_\ell(\pi/2) = 0$, $c_\ell(\pi/2) = 0$ and $s'_\ell(\pi/2) = 0$) provides the resonances in the A_1 (respectively, A_2, B_1 and B_2) representation.

3.3. Asymptotic expansions for Regge poles

In order to determine asymptotic expansions (for $kc \to \infty$) for the Regge poles $b_\ell(kc)$, we have to solve the equation $U^{(1)}(\xi_0, b) = 0$. This will be done by extending the reasoning of subsection 2.2. Hence we need the uniform asymptotic expansion (C9) (limited to the first two leading terms) found by using the Langer–Olver method (see appendix C) which reads, for $\xi = \xi_0$,

$$U^{(1)}(\xi_0, b) = \mathrm{e}^{-\mathrm{i}\pi/2}\zeta^{1/4}\left(b^2 - \cosh^2\xi_0\right)^{-1/4} \times \left\{\mathrm{Ai}\left[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta\right] + B_0(\zeta)\frac{\mathrm{e}^{2\mathrm{i}\pi/3}\mathrm{Ai}'\left[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta\right]}{(kc)^{4/3}}\right\} \times \left[1 + \underset{kc\to\infty}{\mathrm{O}}\left(\frac{1}{(kc)^2}\right)\right]. \tag{50}$$

Here, ζ is given by

$$\tfrac{2}{3}\zeta^{3/2} = -\int_{\operatorname{arccosh} b}^{\xi_0} \left(b^2 - \cosh^2 u\right)^{1/2} \mathrm{d}u \tag{51}$$

and

$$B_0(\zeta) = -\frac{5}{48\zeta^2} + \frac{\sqrt{b^2-1}}{12\sqrt{\zeta}}\left[\frac{\mathrm{i}F\left(\mathrm{i}\xi_0\middle|1/(1-b^2)\right)}{(b^2-1)} - \frac{\mathrm{i}(2b^4-3b^2+1)E\left(\mathrm{i}\xi_0\middle|1/(1-b^2)\right)}{2b^2(b^2-1)^2} + \frac{(14b^4-14b^2+1)\sinh 2\xi_0 + \left(b^2-\frac{1}{2}\right)\sinh 4\xi_0}{8b^2(b^2-1)^2(b^2-\cosh^2\xi_0)^{3/2}}\right] \tag{52}$$

where F and E denote the elliptic integrals of the first and second kinds [2], respectively. To this degree of approximation, the equation $U^{(1)}(\xi_0, b) = 0$ reduces to

$$\frac{\mathrm{Ai}\left[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta\right]}{\mathrm{Ai}'\left[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta\right]} \approx -\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{-4/3}B_0(\zeta). \tag{53}$$

Regge poles $b_\ell(kc)$ can then be sought as the solutions of

$$kc\zeta^{3/2} = -(x_\ell + \delta x)^{3/2} \tag{54}$$

where the x_ℓ ($\ell \in \mathbb{N}^*$) denote, as always, the real negative zeros of the Airy function $\mathrm{Ai}(x)$ and where

$$\delta x = -\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{-4/3}B_0(\zeta). \tag{55}$$

Equation (54) is solved perturbatively, and we obtain, for Regge poles, the following asymptotic expansions:

$$
\begin{aligned}
b_\ell(kc) = \cosh\xi_0 &- \frac{2^{-1/3}\mathrm{e}^{\mathrm{i}\pi/3}(\sinh\xi_0)^{2/3}}{(\cosh\xi_0)^{1/3}} q_{1,0}(x_\ell)(kc)^{-2/3} \\
&+ \frac{2^{-2/3}\mathrm{e}^{2\mathrm{i}\pi/3}}{60(\cosh\xi_0)^{5/3}(\sinh\xi_0)^{2/3}} \left\{q_{2,0}(x_\ell) + q_{2,1}(x_\ell)\cosh 2\xi_0\right\}(kc)^{-4/3} \\
&+ \frac{1}{16\,800(\cosh\xi_0)^3(\sinh\xi_0)^2} \\
&\times \left\{q_{3,0}(x_\ell) + q_{3,1}(x_\ell)\cosh 2\xi_0 + q_{3,2}(x_\ell)\cosh 4\xi_0\right\}(kc)^{-2} \\
&- \frac{2^{-1/3}\mathrm{e}^{\mathrm{i}\pi/3}}{36\,288\,000(\cosh\xi_0)^{13/3}(\sinh\xi_0)^{10/3}} \left\{q_{4,0}(x_\ell) + q_{4,1}(x_\ell)\cosh 2\xi_0\right. \\
&\left.+ q_{4,2}(x_\ell)\cosh 4\xi_0 + q_{4,3}(x_\ell)\cosh 6\xi_0\right\}(kc)^{-8/3} \\
&+ \frac{2^{-2/3}\mathrm{e}^{2\mathrm{i}\pi/3}}{167\,650\,560\,000(\cosh\xi_0)^{17/3}(\sinh\xi_0)^{14/3}} \\
&\times \left\{q_{5,0}(x_\ell) + q_{5,1}(x_\ell)\cosh 2\xi_0 + q_{5,2}(x_\ell)\cosh 4\xi_0\right. \\
&\left.+ q_{5,3}(x_\ell)\cosh 6\xi_0 + q_{5,4}(x_\ell)\cosh 8\xi_0\right\}(kc)^{-10/3} \\
&+ \frac{1}{6457\,651\,200\,000(\cosh\xi_0)^7(\sinh\xi_0)^6} \\
&\times \left\{q_{6,0}(x_\ell) + q_{6,1}(x_\ell)\cosh 2\xi_0 + q_{6,2}(x_\ell)\cosh 4\xi_0\right. \\
&\left.+ q_{6,3}(x_\ell)\cosh 6\xi_0 + q_{6,4}(x_\ell)\cosh 8\xi_0 + q_{6,5}(x_\ell)\cosh 10\xi_0\right\}(kc)^{-4} \\
&+ \underset{kc\to\infty}{\mathrm{O}}\left[(kc)^{-14/3}\right].
\end{aligned} \tag{56}
$$

Here the $q_{i,j}(x_\ell)$ are polynomials of degree i in x_ℓ, given by

$$q_{1,0}(x_\ell) = x_\ell \tag{57a}$$

$$
\begin{aligned}
q_{2,0}(x_\ell) &= 15x_\ell^2 \\
q_{2,1}(x_\ell) &= x_\ell^2
\end{aligned} \tag{57b}
$$

$$
\begin{aligned}
q_{3,0}(x_\ell) &= 570 + 407x_\ell^3 \\
q_{3,1}(x_\ell) &= -980x_\ell^3 \\
q_{3,2}(x_\ell) &= -3(10 + x_\ell^3)
\end{aligned} \tag{57c}
$$

$$
\begin{aligned}
q_{4,0}(x_\ell) &= 90(6840x_\ell + 13\,711x_\ell^4) \\
q_{4,1}(x_\ell) &= -21(119\,880x_\ell + 48\,037x_\ell^4) \\
q_{4,2}(x_\ell) &= -90(360x_\ell - 5641x_\ell^4) \\
q_{4,3}(x_\ell) &= 10\,440x_\ell + 281x_\ell^4
\end{aligned} \tag{57d}
$$

$$
\begin{aligned}
q_{5,0}(x_\ell) &= 3(4557\,365\,100x_\ell^2 + 1125\,295\,351x_\ell^5)\\
q_{5,1}(x_\ell) &= -27\,720(283\,050x_\ell^2 + 266\,977x_\ell^5)\\
q_{5,2}(x_\ell) &= 4(1919\,325\,600x_\ell^2 + 481\,897\,921x_\ell^5)\\
q_{5,3}(x_\ell) &= 9240(8550x_\ell^2 - 60\,997x_\ell^5)\\
q_{5,4}(x_\ell) &= -(6624\,900x_\ell^2 + 73\,769x_\ell^5)
\end{aligned} \tag{57e}
$$

$$
\begin{aligned}
q_{6,0}(x_\ell) &= 5850(318\,120 + 33\,314\,840x_\ell^3 + 15\,814\,949x_\ell^6)\\
q_{6,1}(x_\ell) &= -10(1491\,919\,000 + 63\,787\,753\,000x_\ell^3 + 10\,704\,688\,123x_\ell^6)\\
q_{6,2}(x_\ell) &= -2600(75\,240 - 48\,704\,080x_\ell^3 - 27\,746\,557x_\ell^6)\\
q_{6,3}(x_\ell) &= 35(13\,413\,400 - 2239\,211\,400x_\ell^3 - 382\,250\,049x_\ell^6)\\
q_{6,4}(x_\ell) &= 650(3960 - 886\,040x_\ell^3 + 4013\,887x_\ell^6)\\
q_{6,5}(x_\ell) &= -(1744\,600 - 16\,495\,400x_\ell^3 - 93\,617x_\ell^6).
\end{aligned} \tag{57f}
$$

It should be noted that the asymptotic expansions (25) for the Regge poles $\nu_\ell(ka)$ of the circular cylinder are limiting cases of the expansions (56). Indeed, in the limit $\xi_0 \to \infty$, we have $\cosh\xi_0 \sim \sinh\xi_0 \sim \frac{1}{2}\exp\xi_0$. Then, by putting $ka = (kc/2)\exp\xi_0$ and $\nu_\ell(ka) = kcb_\ell(kc)$ into equation (56) we recover equation (25).

3.4. Asymptotic expansions for resonances

We shall now determine asymptotic expansions for the resonances of the elliptic cylinder. We have to solve the four equations $c'_\ell(\pi/2) = 0$, $s_\ell(\pi/2) = 0$, $c_\ell(\pi/2) = 0$ and $s'_\ell(\pi/2) = 0$, providing the resonances associated, respectively, with the four representations A_1, A_2, B_1 and B_2. With this aim in mind, we use the WKB expansions (D9)–(D12), where we have taken $\eta = \pi/2$ and $b = b_\ell$. The four conditions for resonance then reduce to

$$
\exp\left[2\mathrm{i}\Phi_\ell(kc)\right] = \pm\left[1 - \frac{2Y_1(\pi/2, b_\ell)}{kc} + \underset{kc\to\infty}{\mathrm{O}}\left(\frac{1}{(kc)^2}\right)\right] \tag{58}
$$

where the upper (respectively, the lower) sign corresponds to the A_1 and A_2 resonances (respectively, the B_1 and B_2 resonances). Here,

$$
\Phi_\ell(kc) = kc\int_0^{\pi/2}\left(b_\ell^2 - \cos^2\eta\right)^{1/2}\,\mathrm{d}\eta = kc\sqrt{b_\ell^2 - 1}\,E\left[1/(1-b_\ell^2)\right] \tag{59}
$$

and

$$
Y_1(\pi/2, b_\ell) = \frac{\left(1 - 2b_\ell^2\right)E\left[1/(1-b_\ell^2)\right] + 2b_\ell^2 K\left[1/(1-b_\ell^2)\right]}{24b_\ell^2\sqrt{1-b_\ell^2}} \tag{60}
$$

where $K(1/(1-b_\ell^2))$ and $E(1/(1-b_\ell^2))$ are complete elliptic integrals of the first and second kinds, respectively, [2]. Finally, the conditions for resonance (58) can be written as

$$
\Phi_\ell(kc) = \frac{n\pi}{2} - \frac{Y_1(\pi/2, b_\ell)}{\mathrm{i}kc} + \underset{kc\to\infty}{\mathrm{O}}\left(\frac{1}{(kc)^2}\right) \tag{61}
$$

with $n \in \mathbb{N}^*$. The A_1 and A_2 resonances (respectively, the B_1 and B_2 resonances) correspond to n even (respectively, to n odd).

By taking for $b_\ell(kc)$ its asymptotic expansion (56), we can solve (61) perturbatively, and we obtain

$$\begin{aligned}(kc)_{n,\ell} = &\frac{\pi}{2\widetilde{E}(\xi_0)}n + \frac{e^{i\pi/3}\pi^{1/3}x_\ell \widetilde{K}(\xi_0)}{2^{2/3}\widetilde{E}^{4/3}(\xi_0)\,(\cosh\xi_0\sinh\xi_0)^{1/3}}n^{1/3} \\ &+\frac{e^{2i\pi/3}x_\ell^2\, Q_2(\xi_0,x_\ell)}{2^{1/3}\pi^{1/3}60\widetilde{E}^{5/3}(\xi_0)\,(\cosh\xi_0\sinh\xi_0)^{5/3}}n^{-1/3} \\ &+\frac{Q_3(\xi_0,x_\ell)}{8400\pi\,\widetilde{E}^2(\xi_0)\,(\cosh\xi_0\sinh\xi_0)^3}n^{-1} \\ &+\frac{e^{i\pi/3}x_\ell\, Q_4(\xi_0,x_\ell)}{2^{2/3}\pi^{5/3}4536\,000\widetilde{E}^{7/3}(\xi_0)\,(\cosh\xi_0\sinh\xi_0)^{13/3}}n^{-5/3} \\ &+\frac{e^{2i\pi/3}x_\ell^2 Q_5(\xi_0,x_\ell)}{2^{1/3}\pi^{7/3}41\,912\,640\,000\widetilde{E}^{8/3}(\xi_0)\,(\cosh\xi_0\sinh\xi_0)^{17/3}}n^{-7/3} + \underset{n\to+\infty}{O}\left(n^{-3}\right)\end{aligned} \tag{62}$$

with

$$Q_2(\xi_0,x_\ell) = 15\sinh 2\xi_0\widetilde{E}^2(\xi_0) - 16\cosh 2\xi_0\widetilde{E}(\xi_0)\widetilde{K}(\xi_0) + 10\sinh 2\xi_0\widetilde{K}^2(\xi_0) \tag{63a}$$

$$\begin{aligned}Q_3(\xi_0,x_\ell) = &-(175+140x_\ell^3)\sinh 4\xi_0\widetilde{E}^3(\xi_0) \\ &+\left[-745+433x_\ell^3+(205+143x_\ell^3)\cosh 4\xi_0\right]\widetilde{K}(\xi_0)\widetilde{E}^2(\xi_0)\end{aligned} \tag{63b}$$

$$\begin{aligned}Q_4(\xi_0,x_\ell) = &-45\left[21\,540+3631x_\ell^3+\left(1740-239x_\ell^3\right)\cosh 4\xi_0\right]\sinh 2\xi_0\widetilde{E}^4(\xi_0) \\ &-4\left[278\,415-95\,054x_\ell^3-\left(37\,935-3406x_\ell^3\right)\cosh 4\xi_0\right]\cosh 2\xi_0\widetilde{E}^3(\xi_0)\widetilde{K}(\xi_0) \\ &+60\left[4470-2227x_\ell^3-\left(1230-563x_\ell^3\right)\cosh 4\xi_0\right]\sinh 2\xi_0\widetilde{E}^2(\xi_0)\widetilde{K}^2(\xi_0) \\ &-67\,200x_\ell^3\cosh 2\xi_0\,(\sinh 2\xi_0)^2\,\widetilde{E}(\xi_0)\widetilde{K}^3(\xi_0)+14\,000x_\ell^3\,(\sinh 2\xi_0)^3\,\widetilde{K}^4(\xi_0)\end{aligned} \tag{63c}$$

$$\begin{aligned}Q_5(\xi_0,x_\ell) = &-4620\left[2000\,025+127\,304x_\ell^3+23\left(225-632x_\ell^3\right)\cosh 4\xi_0\right]\sinh 4\xi_0\widetilde{E}^5(\xi_0) \\ &+\Big[-3\left(4569\,434\,850-493\,885\,289x_\ell^3\right) \\ &+4\left(20\,901\,150+314\,044\,919x_\ell^3\right)\cosh 4\xi_0 \\ &+\left(49\,071\,150-77\,024\,791x_\ell^3\right)\cosh 8\xi_0\Big]\widetilde{E}^4(\xi_0)\widetilde{K}(\xi_0) \\ &+6160\left[425\,925-121\,456x_\ell^3-\left(22\,725-21\,104x_\ell^3\right)\cosh 4\xi_0\right] \\ &\times\sinh 4\xi_0\widetilde{E}^3(\xi_0)\widetilde{K}^2(\xi_0) \\ &+184\,800\left[2235-753x_\ell^3\left(615-817x_\ell^3\right)\cosh 4\xi_0\right](\sinh 2\xi_0)^2\,\widetilde{E}^2(\xi_0)\widetilde{K}^3(\xi_0) \\ &+137\,984\,000x_\ell^3\cosh 2\xi_0\,(\sinh 2\xi_0)^3\,\widetilde{E}(\xi_0)\widetilde{K}^4(\xi_0) \\ &-21\,560\,000x_\ell^3\,(\sinh 2\xi_0)^4\,\widetilde{K}^5(\xi_0).\end{aligned} \tag{63d}$$

In the previous formula, we have defined

$$\widetilde{K}(\xi_0) = \cosh\xi_0 K\left(-\frac{1}{\sinh^2\xi_0}\right) \qquad \text{and} \qquad \widetilde{E}(\xi_0) = \sinh\xi_0 E\left(-\frac{1}{\sinh^2\xi_0}\right). \tag{64}$$

By considering the limit $\xi_0 \to +\infty$ and then by putting $(ka)_{n,\ell} = ((kc)_{n,\ell}/2)\exp\xi_0$ into (62), we recover expansions (27) which provide the resonances of the circular cylinder.

3.5. Exponentially improved asymptotic expansions for resonances

The splitting up of resonances, emphasized in subsection 3.1, does not appear in the expansions (62). In fact, even by considering higher orders in the WKB expansions of the functions $c(\eta, b)$ and $s(\eta, b)$, it is not possible to display such a phenomenon. Perturbation theory fails to provide this splitting because it corresponds to an exponentially small term which lies beyond all orders of the WKB expansions and can be captured only by taking Stokes' phenomenon carefully into account [23]. With this aim in mind, the hyperasymptotic treatment, developed by Dingle [22], and more recently by Berry and Howls [24], and which is based on Borel summation and on Dingle's resurgence formula, could be helpful. However, we prefer to make use of the uniform asymptotic expansions for the functions $c(\eta, b)$, $s(\eta, b)$, $c'(\eta, b)$ and $s'(\eta, b)$. Furthermore, in order to simplify the notation, we intensively use the symbol $\approx$, instead of rigorous mathematical symbols.

The conditions for resonance $c'_\ell(\pi/2) = 0$, $s_\ell(\pi/2) = 0$, $c_\ell(\pi/2) = 0$ and $s'_\ell(\pi/2) = 0$, written in terms of the uniform asymptotic expansions (C30)–(C33), reduce, respectively, to

$$\mathrm{Ai}'\left((kc)^{2/3}\zeta(0)\right)\mathrm{Ai}'\left(\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(\pi/2)\right) \approx \mathrm{Ai}'\left(\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(0)\right)\mathrm{Ai}'\left((kc)^{2/3}\zeta(\pi/2)\right) \tag{65a}$$

$$\mathrm{Ai}\left((kc)^{2/3}\zeta(0)\right)\mathrm{Ai}\left(\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(\pi/2)\right) \approx \mathrm{Ai}\left(\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(0)\right)\mathrm{Ai}\left((kc)^{2/3}\zeta(\pi/2)\right) \tag{65b}$$

$$\mathrm{Ai}'\left((kc)^{2/3}\zeta(0)\right)\mathrm{Ai}\left(\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(\pi/2)\right) \approx \mathrm{e}^{2\mathrm{i}\pi/3}\mathrm{Ai}'\left(\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(0)\right)\mathrm{Ai}\left((kc)^{2/3}\zeta(\pi/2)\right) \tag{65c}$$

$$\mathrm{Ai}\left((kc)^{2/3}\zeta(0)\right)\mathrm{e}^{2\mathrm{i}\pi/3}\mathrm{Ai}'\left(\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(\pi/2)\right) \approx \mathrm{Ai}\left(\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(0)\right)\mathrm{Ai}'\left((kc)^{2/3}\zeta(\pi/2)\right). \tag{65d}$$

In the previous equations, $\zeta(\eta)$ is defined by the relation (C20). In order to express these equations in a simpler way, we can use the asymptotic behaviour of the Airy functions. For $z = (kc)^{2/3}\zeta(0)$, $z = (kc)^{2/3}\zeta(\pi/2)$ and $z = \mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(\pi/2)$, we shall write

$$\mathrm{Ai}(z) \approx \frac{1}{2\sqrt{\pi}} z^{-1/4} \exp\left(-\tfrac{2}{3}z^{3/2}\right) \tag{66}$$

and

$$\mathrm{Ai}'(z) \approx -\frac{1}{2\sqrt{\pi}} z^{1/4} \exp\left(-\tfrac{2}{3}z^{3/2}\right) \tag{67}$$

because $|\arg z| < 2\pi/3$, while for $z = \mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(0)$, we shall use

$$\mathrm{Ai}(z) \approx \frac{1}{2\sqrt{\pi}} z^{-1/4} \left[\exp\left(-\tfrac{2}{3}z^{3/2}\right) + \mathrm{i}\exp\left(\tfrac{2}{3}z^{3/2}\right)\right] \tag{68}$$

and

$$\mathrm{Ai}'(z) \approx -\frac{1}{2\sqrt{\pi}} z^{1/4} \left[\exp\left(-\tfrac{2}{3}z^{3/2}\right) - \mathrm{i}\exp\left(\tfrac{2}{3}z^{3/2}\right)\right] \tag{69}$$

because z lies beyond the Stokes line $\arg z = 2\pi/3$. Now we substitute the asymptotic expansions (66)–(69) in equations (65). By noting that the function $\Phi_\ell(kc)$, already defined by (59), can also be expressed from (C20) as $\Phi_\ell(kc) = -\frac{2}{3}\mathrm{i}kc[\zeta(0)^{3/2} - \zeta(\pi/2)^{3/2}]$, we obtain

$$\exp[2\mathrm{i}\phi_\ell(kc)] \approx \pm 1 + \mathrm{i}\exp\left[-\tfrac{4}{3}kc\zeta(\pi/2)^{3/2}\right] \tag{70a}$$

where the upper (respectively, the lower) sign corresponds to the A_1 (respectively, the B_1) equation, and

$$\exp\left[2\mathrm{i}\phi_\ell(kc)\right] \approx \pm 1 - \mathrm{i}\exp\left[-\tfrac{4}{3}kc\zeta(\pi/2)^{3/2}\right] \tag{70b}$$

where the upper (respectively, the lower) sign corresponds to the A_2 (respectively, the B_2) equation.

Finally, from equations (70), the condition for resonance (61) is exponentially improved and reads

$$\Phi_\ell(kc) = \frac{n\pi}{2} - \frac{Y_1(\pi/2, b_\ell)}{\mathrm{i}kc} + \underbrace{\cdots\cdots\cdots\cdots\cdots\cdots\cdots}_{\text{higher orders in } 1/(kc)} \pm \underbrace{\tfrac{1}{2}(-1)^n \exp\left[-\tfrac{4}{3}kc\zeta(\pi/2)^{3/2}\right]}_{\text{term lying beyond all orders}}. \tag{71}$$

with $n \in \mathbb{N}^*$.

Now, we have to solve equation (71). Let us denote its solutions by $(\widetilde{kc})_{n,\ell}$. Each $(\widetilde{kc})_{n,\ell}$ can be sought close to the corresponding solution $(kc)_{n,\ell}$ of equation (61). By expanding $\Phi_\ell\left((\widetilde{kc})_{n,\ell}\right)$ in Taylor series about $(kc)_{n,\ell}$, we obtain

$$(\widetilde{kc})_{n,\ell} = (kc)_{n,\ell} \pm \frac{(-1)^n}{2\,(\mathrm{d}\Phi_\ell/\mathrm{d}kc)_{(kc)_{n,\ell}}} \exp\left[-\tfrac{4}{3}(kc)_{n,\ell}\zeta(\pi/2)^{3/2}\right]. \tag{72}$$

The A_1 (respectively, the A_2) resonances correspond to n even and to the upper (respectively, the lower) sign in equations (71) and (72). The B_1 (respectively, the B_2) resonances correspond to n odd and to the upper (respectively, the lower) sign in equations (71) and (72). Since b_ℓ depends on kc, the differentiation of (59) can be carried out and leads to a series expansion for $\mathrm{d}\Phi_\ell/\mathrm{d}kc$. However, for numerical evaluations, the leading term

$$\left(\frac{\mathrm{d}\Phi_\ell}{\mathrm{d}kc}\right)_{(kc)_{n,\ell}} \approx \sinh\xi_0 E\left(-\frac{1}{\sinh^2\xi_0}\right) = \widetilde{E}(\xi_0) \tag{73}$$

provides sufficiently accurate results in most configurations for ξ_0 and $(kc)_{n,\ell}$.

Table 3. A_1 and A_2 resonances for $\xi_0 = 1$.

n, ℓ	$kc^{(A_1)}_{n,\ell}$	$kc^{(A_2)}_{n,\ell}$	
8, 1	4.327 498 712 − 2.019 572 915i	4.327 498 538 − 2.019 572 709i	Exact
	4.327 463 148 − 2.019 171 483i	4.327 462 990 − 2.019 171 324i	Asymptotic
10, 1	5.702 157 958 − 2.222 922 078i	5.702 157 955 − 2.222 922 069i	Exact
	5.702 122 958 − 2.222 673 723i	5.702 122 955 − 2.222 673 713i	Asymptotic

Table 4. B_1 and B_2 resonances for $\xi_0 = 1$.

n, ℓ	$kc^{(B_1)}_{n,\ell}$	$kc^{(B_2)}_{n,\ell}$	
7, 1	3.644 907 129 − 1.903 103 239i	3.644 906 212 − 1.903 102 585i	Exact
	3.644 878 040 − 1.902 570 682i	3.644 877 138 − 1.902 570 130i	Asymptotic
9, 1	5.013 419 221 − 2.125 489 923i	5.013 419 197 − 2.125 489 881i	Exact
	5.013 382 779 − 2.125 178 031i	5.013 382 754 − 2.125 177 990i	Asymptotic

Table 5. Splitting of resonances for $\xi_0 = 1$.

	Exact		Asymptotic	
n, ℓ	$\ln\left|\mathrm{Re}(\delta kc_{n,\ell})\right|$	$\ln\left|\mathrm{Im}(\delta kc_{n,\ell})\right|$	$\ln\left|\mathrm{Re}(\delta kc_{n,\ell})\right|$	$\ln\left|\mathrm{Im}(\delta kc_{n,\ell})\right|$
7, 1	−13.902	−14.240	−13.919	−14.409
8, 1	−15.564	−15.395	−15.664	−15.651
9, 1	−17.550	−16.967	−17.497	−17.017
10, 1	−19.575	−18.426	−19.480	−18.453

The asymptotic expansion (72) gives an even better approximation when n and ξ_0 are large. In tables 3 and 4 we present the exact values and exponentially improved asymptotic approximations of resonances $(\widetilde{kc})_{n,\ell}$ for $\xi_0 = 1$, $\ell = 1$ and for various n. A good agreement is found, even though a discrepancy appears in the last decimal places. It should be noted that this discrepancy is due to the perturbative part of the asymptotic expansion. A better agreement could be obtained by improving the asymptotic expansion (62) (by looking for higher-order terms in $n^{-3}, n^{-11/3}, n^{-13/3}, \ldots$). In table 5, the asymptotic splitting

$$\delta kc_{n,\ell} = \frac{(-1)^n}{(\mathrm{d}\Phi_\ell/\mathrm{d}kc)_{(kc)_{n,\ell}}} \exp\left[-\tfrac{4}{3}(kc)_{n,\ell}\zeta(\pi/2)^{3/2}\right] \tag{74}$$

is compared with the exact one and a very good agreement is displayed.

4. Conclusion and perspectives

This paper could be extended in several directions.

- The uniform asymptotic expansions obtained for Mathieu and modified Mathieu functions could be used to construct uniform approximations for the scattering amplitude in terms of Fock or generalized Fock functions [29]. With this aim in mind, the uniform theory of diffraction (see [30] for a pedagogical introduction) developed by Pathak [31] for convex cylinders could be very useful.
- The method developed here could be extended to determine semiclassically the eigenvalues of the elliptic quantum billiard. Such considerations would greatly improve the results obtained from EBK quantization [15–17]. Furthermore, they would make the numerical search for the exact eigenvalues easier by providing starting points for the numerical algorithms.
- More generally, our method could be applied in the context of three-dimensional billiards or scatterers of spheroidal and related shapes. In these cases, the splitting up of resonances is a little more complex because it is associated with the breaking of the $O(3)$ symmetry of the sphere.

Furthermore, it would be also interesting to observe experimentally the splitting up of resonances in acoustics or in electromagnetism by studying scattering of ultrasonic waves from elastic elliptic cylinders, or scattering of microwaves from conducting or dielectric elliptic cylinders.

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Appendix A. Orthonormalization relations for the radial modes

The construction of the diffracted Green functions (for the circular and elliptic cylinders) involves the orthogonality properties of the radial modes. These properties are obtained by using the radial parts of the Helmholtz equation and the asymptotic behaviour of the outgoing solutions.

Appendix A.1. The case of the circular cylinder

We must evaluate the integral

$$\int_a^{+\infty} H_{\nu_\ell}^{(1)}(k\rho)\, H_{\nu_m}^{(1)}(k\rho)\, \frac{\mathrm{d}\rho}{\rho} \tag{A1}$$

where ν_ℓ and ν_m are two Regge poles. Let us first consider the more general integral

$$\int_a^{+\infty} H_{\nu_\ell}^{(1)}(k\rho)\, H_{\nu}^{(1)}(k\rho)\, \frac{\mathrm{d}\rho}{\rho} \tag{A2}$$

where ν_ℓ is a Regge pole while ν is arbitrary. From Bessel's equation (7), we obtain

$$\int_a^{+\infty} H_{\nu_\ell}^{(1)}(k\rho) H_{\nu}^{(1)}(k\rho)\, \frac{\mathrm{d}\rho}{\rho} = \frac{1}{\nu_\ell^2 - \nu^2}\left[\rho H_{\nu}^{(1)}(k\rho)\, \frac{\mathrm{d}H_{\nu_\ell}^{(1)}}{\mathrm{d}\rho} - \rho H_{\nu_\ell}^{(1)}(k\rho)\, \frac{\mathrm{d}H_{\nu}^{(1)}}{\mathrm{d}\rho}\right]_a^{+\infty}. \tag{A3}$$

By using the asymptotic behaviour of the Hankel function of the first kind, it can be shown that the contributions for $\rho \to +\infty$ cancel each other. Then, since $H_{\nu_\ell}^{(1)}(ka) = 0$, it yields

$$\int_a^{+\infty} H_{\nu_\ell}^{(1)}(k\rho) H_{\nu}^{(1)}(k\rho)\, \frac{\mathrm{d}\rho}{\rho} = \frac{a H_{\nu}^{(1)}(ka)}{\nu^2 - \nu_\ell^2} \left(\frac{\mathrm{d}H_{\nu_\ell}^{(1)}(k\rho)}{\mathrm{d}\rho}\right)_{\rho=a}. \tag{A4}$$

When $\nu = \nu_m$ is a Regge pole, the right-hand side of (A4) simplifies. If $m \neq \ell$, it vanishes. If $m = \ell$, the limit $\nu \to \nu_\ell$ in (A4) can be carried out by writing $\nu = \nu_\ell + \epsilon$ with $\epsilon \to 0$. It follows that

$$\int_a^{+\infty} \left[H_{\nu_\ell}^{(1)}(k\rho)\right]^2 \frac{\mathrm{d}\rho}{\rho} = \lim_{\epsilon\to 0} \frac{a H_{\nu_\ell+\epsilon}^{(1)}(ka)}{2\epsilon \nu_\ell} \left(\frac{\mathrm{d}H_{\nu_\ell}^{(1)}(k\rho)}{\mathrm{d}\rho}\right)_{\rho=a}. \tag{A5}$$

By making use of the expansion

$$H_{\nu_\ell+\epsilon}^{(1)}(ka) = \underbrace{H_{\nu_\ell}^{(1)}(ka)}_{=0} + \epsilon \left(\frac{\partial H_{\nu}^{(1)}(ka)}{\partial \nu}\right)_{\nu=\nu_\ell} + \underset{\epsilon\to 0}{\mathrm{O}}(\epsilon^2) \tag{A6}$$

we can write

$$\int_a^{+\infty} \left[H_{\nu_\ell}^{(1)}(k\rho)\right]^2 \frac{\mathrm{d}\rho}{\rho} = \frac{a}{2\nu_\ell} \left(\frac{\partial H_{\nu}^{(1)}(ka)}{\partial \nu}\right)_{\nu=\nu_\ell} \left(\frac{\mathrm{d}H_{\nu_\ell}^{(1)}(k\rho)}{\mathrm{d}\rho}\right)_{\rho=a}. \tag{A7}$$

Finally, we combine the cases $\ell = m$ and $\ell \neq m$ in the orthonormalization relation

$$\int_a^{\infty} H_{\nu_m}^{(1)}(k\rho) H_{\nu_\ell}^{(1)}(k\rho)\, \frac{\mathrm{d}\rho}{\rho} = N_\ell(ka)\delta_{\ell m} \tag{A8}$$

where

$$N_\ell(ka) = \frac{a}{2\nu_\ell} \left(\frac{\partial H_{\nu}^{(1)}(ka)}{\partial \nu}\right)_{\nu=\nu_\ell} \left(\frac{\mathrm{d}H_{\nu_\ell}^{(1)}(k\rho)}{\mathrm{d}\rho}\right)_{\rho=a}. \tag{A9}$$

An alternative expression for (A9) can be found by using the Wronskian [2]

$$W\left[H_\nu^{(1)}(x), H_\nu^{(2)}(x)\right] = H_\nu^{(1)}(x)\,\frac{\mathrm{d}H_\nu^{(2)}(x)}{\mathrm{d}x} - \frac{\mathrm{d}H_\nu^{(1)}(x)}{\mathrm{d}x}\,H_\nu^{(2)}(x) = -\frac{4\mathrm{i}}{\pi x}. \tag{A10}$$

Indeed, if $x = ka$ and ν_ℓ is a Regge pole, we have the relation

$$aH_{\nu_\ell}^{(2)}(ka)\left(\frac{\mathrm{d}H_{\nu_\ell}^{(1)}(k\rho)}{\mathrm{d}\rho}\right)_{\rho=a} = \frac{4\mathrm{i}}{\pi} \tag{A11}$$

thus

$$N_\ell(ka) = \frac{2\mathrm{i}}{\pi\,\nu_\ell H_{\nu_\ell}^{(2)}(ka)}\left(\frac{\partial H_\nu^{(1)}(ka)}{\partial\nu}\right)_{\nu=\nu_\ell}. \tag{A12}$$

Appendix A.2. The case of the elliptic cylinder

Consider now the orthonormalization relation for the solutions $U_\ell^{(1)}(\xi)$ of the radial problem in the case of the elliptic cylinder. The calculation, involving the modified Mathieu equation (35) as well as the outgoing behaviour (C14), is very similar to the previous one and the final result is simply given by

$$\int_{\xi_0}^{+\infty} U_\ell^{(1)}(\xi)\,U_m^{(1)}(\xi)\,\mathrm{d}\xi = N_\ell(\xi_0)\delta_{\ell m} \tag{A13}$$

with

$$N_\ell(\xi_0) = \frac{1}{2(kc)^2 b_\ell}\left(\frac{\mathrm{d}U_\ell^{(1)}}{\mathrm{d}\xi}\right)_{\xi=\xi_0}\left(\frac{\partial U^{(1)}(\xi_0, b)}{\partial b}\right)_{b=b_\ell}. \tag{A14}$$

An alternative expression for $N_\ell(\xi_0)$ involving the incoming solution $U_\ell^{(2)}(\xi)$ of the modified Mathieu equation can be found. Indeed, let us consider the Wronskian

$$W\left[U^{(1)}(\xi), U^{(2)}(\xi)\right] = U^{(1)}(\xi)\,\frac{\mathrm{d}U^{(2)}}{\mathrm{d}\xi} - \frac{\mathrm{d}U^{(1)}}{\mathrm{d}\xi}\,U^{(2)}(\xi). \tag{A15}$$

From equation (35), we have $\mathrm{d}W\left[U^{(1)}(\xi), U^{(2)}(\xi)\right]/\mathrm{d}\xi = 0$. The Wronskian is therefore a constant which can be determined by considering the limiting case $\xi \to \infty$. From (C14) and (C15) and by taking into account the fact that $\lim_{\xi\to\infty}\tanh\xi = 1$, we obtain

$$W\left[U^{(1)}(\xi), U^{(2)}(\xi)\right] = -\frac{\mathrm{e}^{-\mathrm{i}\pi/6}(kc)^{2/3}}{2\pi}. \tag{A16}$$

If $\xi = \xi_0$ and $b = b_\ell$, equation (A15) then reduces to

$$U_\ell^{(2)}(\xi_0)\left(\frac{\mathrm{d}U_\ell^{(1)}}{\mathrm{d}\xi}\right)_{\xi=\xi_0} = \frac{\mathrm{e}^{-\mathrm{i}\pi/6}(kc)^{2/3}}{2\pi}. \tag{A17}$$

From (A14) and (A17), we obtain

$$N_\ell(\xi_0) = \frac{\mathrm{e}^{-\mathrm{i}\pi/6}}{4\pi(kc)^{4/3}b_\ell U_\ell^{(2)}(\xi_0)}\left(\frac{\partial U^{(1)}(\xi_0)}{\partial b}\right)_{b=b_\ell}. \tag{A18}$$

Appendix B. Some symmetry properties of the solutions of the ordinary Mathieu equation

In order to expand the diffracted Green function over the four irreducible representations A_1, A_2, B_1 and B_2, we need certain symmetry properties of the solutions of the Mathieu equation. They are obtained from the symmetries of the Mathieu equation.

Let us consider the two linearly independent solutions $c(\eta, b)$ and $s(\eta, b)$ of the ordinary Mathieu equation (36), satisfying the initial conditions

$$\begin{aligned} c(0,b) &= 1 \qquad c'(0,b) = 0 \\ s(0,b) &= 0 \qquad s'(0,b) = 1 \end{aligned} \tag{B1}$$

and their Wronskian

$$W[c(\eta,b), s(\eta,b)] = c(\eta,b)\, s'(\eta,b) - c'(\eta,b)\, s(\eta,b). \tag{B2}$$

From equation (36), we obtain $\mathrm{d}W\,[c(\eta,b), s(\eta,b)]\,/\mathrm{d}\eta = 0$, so the Wronskian is a constant. It is determined from (B1) and reduces to

$$W\,[c(\eta,b), s(\eta,b)] = 1. \tag{B3}$$

Since the Mathieu equation (36) is invariant under the transformations $\eta \to -\eta$, $\eta \to \pi+\eta$ and $\eta \to \pi - \eta$, the functions $c(-\eta, b)$, $c(\pi+\eta, b)$ and $c(\pi-\eta, b)$ as well as the functions $s(-\eta, b)$, $s(\pi+\eta, b)$ and $s(\pi-\eta, b)$ are also solutions of (36), and hence can be written as linear combinations of $c(\eta, b)$ and $s(\eta, b)$. By differentiating with respect to η these linear combinations and then by taking into account (B1), we establish the following properties:

$$\begin{aligned} c(\pi \pm \eta, b) &= c(\pi,b)c(\eta,b) \pm c'(\pi,b)s(\eta,b) \\ s(\pi \pm \eta, b) &= s(\pi,b)c(\eta,b) \pm s'(\pi,b)s(\eta,b) \\ c'(\pi \pm \eta, b) &= c'(\pi,b)s'(\eta,b) \pm c(\pi,b)c'(\eta,b) \\ s'(\pi \pm \eta, b) &= s'(\pi,b)s'(\eta,b) \pm s(\pi,b)c'(\eta,b). \end{aligned} \tag{B4}$$

By substituting the constant Wronskian's property (B3) into (B4) we obtain, among other relations,

$$\begin{aligned} c(\pi,b) = s'(\pi,b) &= c(\pi/2,b)s'(\pi/2,b) + c'(\pi/2,b)s(\pi/2,b) \\ &= 1 + 2c'(\pi/2,b)s(\pi/2,b) = 2c(\pi/2,b)s'(\pi/2,b) - 1 \\ c'(\pi,b) &= 2c'(\pi/2,b)c(\pi/2,b) \\ s(\pi,b) &= 2s(\pi/2,b)s'(\pi/2,b). \end{aligned} \tag{B5}$$

Appendix C. Uniform asymptotic expansions for the functions arising in the theory of diffraction by an elliptic cylinder

Uniform asymptotic expansions for the functions involved in the diffracted Green function of the elliptic cylinder permits us to determine asymptotic expansions for Regge poles, and to capture the exponentially small terms lying beyond all orders in the asymptotic expansions for resonances. The Langer–Olver method [25] is recalled here briefly, and then applied to Mathieu and modified Mathieu equations.

Appendix C.1. The Langer–Olver method: generalities

Any solution of the equation

$$\frac{d^2W}{dz^2} = \left[\lambda^2 p(z) + q(z)\right] W(z) \tag{C1}$$

where λ is a large parameter, can be represented by the series

$$W(z) = e^{-i\pi/2}\zeta^{1/4}\,[p(z)]^{-1/4}\left\{P(\zeta)\sum_{s=0}^{\infty}\frac{A_s(\zeta)}{\lambda^{2s}} + \frac{dP}{d\zeta}\sum_{s=0}^{\infty}\frac{B_s(\zeta)}{\lambda^{2(s+1)}}\right\}. \tag{C2}$$

If z^* is a simple zero of the function $p(z)$, the variable ζ is defined from

$$\tfrac{2}{3}\zeta^{3/2} = -\int_{z^*}^{z}[p(u)]^{1/2}\,du \tag{C3}$$

and $P(\zeta)$ is a solution of the Airy equation

$$\frac{d^2P}{d\zeta^2} = \lambda^2\zeta P(\zeta). \tag{C4}$$

The sequences of coefficients $A_s(\zeta)$ and $B_s(\zeta)$ are defined, up to integration constants, by the relations

$$A_0(\zeta) = 1 \tag{C5}$$

$$B_s(\zeta) = \tfrac{1}{2}\zeta^{-1/2}\int_0^{\zeta} t^{-1/2}\left[f(t)A_s(t) - \frac{d^2A_s}{dt^2}\right]dt \tag{C6}$$

$$A_{s+1}(\zeta) = -\frac{1}{2}\frac{dB_s}{d\zeta} + \frac{1}{2}\int_0^{\zeta} f(t)B_s(t)\,dt \tag{C7}$$

with

$$f(\zeta) = \frac{5}{16\zeta^2} - \frac{5\zeta}{16}\frac{\left[p'(z)\right]^2}{[p(z)]^3} + \frac{\zeta}{4}\frac{p''(z)}{[p(z)]^2} + \frac{\zeta q(z)}{p(z)}. \tag{C8}$$

Appendix C.2. Application of the Langer–Olver method to the solutions of the modified Mathieu equation

The modified Mathieu equation (35) can be obviously written in the form (C1) by taking $\lambda = kc$, $p(\xi) = b^2 - \cosh^2\xi$ and $q(\xi) = 0$. Consequently, for large kc, we can obtain two linearly independent solutions of (35) such as

$$U^{(1)}(\xi) = e^{-i\pi/2}\zeta^{1/4}\left(b^2 - \cosh^2\xi\right)^{-1/4} \times\left\{\mathrm{Ai}\left[e^{2i\pi/3}(kc)^{2/3}\zeta\right]\sum_{s=0}^{\infty}\frac{A_s(\zeta)}{(kc)^{2s}} + \frac{e^{2i\pi/3}\mathrm{Ai}'\left[e^{2i\pi/3}(kc)^{2/3}\zeta\right]}{(kc)^{4/3}}\sum_{s=0}^{\infty}\frac{B_s(\zeta)}{(kc)^{2s}}\right\} \tag{C9}$$

and

$$U^{(2)}(\xi) = e^{i\pi/2}\zeta^{1/4}\left(b^2 - \cosh^2\xi\right)^{-1/4} \times\left\{\mathrm{Ai}\left[e^{-2i\pi/3}(kc)^{2/3}\zeta\right]\sum_{s=0}^{\infty}\frac{A_s(\zeta)}{(kc)^{2s}} + \frac{e^{-2i\pi/3}\mathrm{Ai}'\left[e^{-2i\pi/3}(kc)^{2/3}\zeta\right]}{(kc)^{4/3}}\sum_{s=0}^{\infty}\frac{B_s(\zeta)}{(kc)^{2s}}\right\} \tag{C10}$$

where ζ is defined by

$$\tfrac{2}{3}\zeta^{3/2} = -\int_{\operatorname{arccosh} b}^{\xi} \left(b^2 - \cosh^2 u\right)^{1/2} \, \mathrm{d}u. \tag{C11}$$

By using the asymptotic behaviour of the Airy function [2] for $\xi \to +\infty$ and the fact that

$$\tfrac{2}{3}\zeta^{3/2} \underset{\xi\to+\infty}{\sim} \mathrm{i}\cosh\xi - \mathrm{i}\tfrac{1}{2}\pi\left(b^2 - \tfrac{1}{2}\right)^{1/2} \tag{C12}$$

we can show that $U^{(1)}$ is the outgoing solution of the modified Mathieu equation, while $U^{(2)}$ is the incoming one. Furthermore, by letting

$$\nu = kc\left(b^2 - \tfrac{1}{2}\right)^{1/2} \tag{C13}$$

as in (39), we obtain the first-order approximations

$$U^{(1)}(\xi) \underset{\xi\to+\infty}{\sim} \frac{(kc)^{-1/6}}{2\sqrt{\pi}} \mathrm{e}^{-\mathrm{i}\pi/6} (\cosh\xi)^{-1/2} \exp\left[\mathrm{i}(kc\cosh\xi - \nu\pi/2 - \pi/4)\right] \tag{C14}$$

for the outgoing function, and

$$U^{(2)}(\xi) \underset{\xi\to+\infty}{\sim} \frac{(kc)^{-1/6}}{2\sqrt{\pi}} \mathrm{e}^{\mathrm{i}\pi/6} (\cosh\xi)^{-1/2} \exp\left[-\mathrm{i}(kc\cosh\xi - \nu\pi/2 - \pi/4)\right] \tag{C15}$$

for the incoming one.

The coefficients $A_s(\zeta)$ and $B_s(\zeta)$ can be determined from the recurrence relations (C5)–(C7). In fact, only $A_0(\zeta)$ and $B_0(\zeta)$ are needed in our work. The function $f(\zeta)$ defined by (C8) is given here by

$$f(\zeta) = \frac{5}{16\zeta^2} - \frac{\zeta}{2}\left[\frac{5}{2}\frac{\cosh^2\xi\sinh^2\xi}{(b^2 - \cosh^2\xi)^3} + \frac{\cosh^2\xi + \sinh^2\xi}{(b^2 - \cosh^2\xi)^2}\right] \tag{C16}$$

and permits us to evaluate $B_0(\zeta)$ from the recurrence relation (C6) by using the relation $(\mathrm{d}\xi/\mathrm{d}\zeta)\,[p(\xi)]^{1/2} = -\zeta^{1/2}$ which follows from differentiation of (C3). We find

$$B_0(\zeta) = -\frac{5}{48\zeta^2} + \frac{\sqrt{b^2-1}}{12\sqrt{\zeta}}\left[\frac{\mathrm{i}F\left(\mathrm{i}\xi\,\middle|\,1/(1-b^2)\right)}{(b^2-1)} - \frac{\mathrm{i}(2b^4 - 3b^2 + 1)E\left(\mathrm{i}\xi\,\middle|\,(1/1-b^2)\right)}{2b^2(b^2-1)^2}\right.$$
$$\left. + \frac{(14b^4 - 14b^2 + 1)\sinh 2\xi + \left(b^2 - \frac{1}{2}\right)\sinh 4\xi}{8b^2(b^2-1)^2(b^2-\cosh^2\xi)^{3/2}}\right] \tag{C17}$$

where F and E denote the elliptic integrals of the first and second kinds, respectively, [2].

Appendix C.3. Application of the Langer–Olver method to the ordinary Mathieu equation

The ordinary Mathieu equation (36) can be written in the form (C1) simply by taking $\lambda = kc$, $p(\eta) = -(b^2 - \cos^2\eta)$ and $q(\eta) = 0$. Thus, for large kc, the Langer–Olver method permits us to obtain two linearly independent solutions of (36) in the form

$$V_-(\eta, b) = \mathrm{e}^{-\mathrm{i}\pi/2}\zeta^{1/4}\left(b^2 - \cos^2\eta\right)^{-1/4}$$
$$\times\left\{\mathrm{Ai}\left[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta\right]\sum_{s=0}^{\infty}\frac{A_s(\zeta)}{(kc)^{2s}} + \frac{\mathrm{e}^{2\mathrm{i}\pi/3}\mathrm{Ai}'\left[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta\right]}{(kc)^{4/3}}\sum_{s=0}^{\infty}\frac{B_s(\zeta)}{(kc)^{2s}}\right\} \tag{C18}$$

and

$$V_+(\eta, b) = \mathrm{e}^{-\mathrm{i}\pi/2}\zeta^{1/4}\left(b^2 - \cos^2\eta\right)^{-1/4}$$
$$\times\left\{\mathrm{Ai}\left[(kc)^{2/3}\zeta\right]\sum_{s=0}^{\infty}\frac{A_s(\zeta)}{(kc)^{2s}} + \frac{\mathrm{Ai}'\left[(kc)^{2/3}\zeta\right]}{(kc)^{4/3}}\sum_{s=0}^{\infty}\frac{B_s(\zeta)}{(kc)^{2s}}\right\} \quad \text{(C19)}$$

where ζ is defined by

$$\tfrac{2}{3}\left[\zeta(\eta)\right]^{3/2} = -\mathrm{i}\int_{\arccos b}^{\eta}\left(b^2 - \cos^2 v\right)^{1/2}\,\mathrm{d}v. \quad \text{(C20)}$$

The Wronskian

$$W\left[V_-(\eta), V_+(\eta)\right] = V_-(\eta)V_+'(\eta) - V_-'(\eta)V_+(\eta) \quad \text{(C21)}$$

does not depend on η. Indeed, from equation (36), we obtain $\mathrm{d}W\left[V_-(\eta), V_+(\eta)\right]/\mathrm{d}\eta = 0$. Furthermore, it can be evaluated by taking the limit $\eta \to \arccos b$ and then by using $W\left[\mathrm{Ai}\left(\mathrm{e}^{2\mathrm{i}\pi/3}z\right), \mathrm{Ai}(z)\right] = -\mathrm{e}^{-\mathrm{i}\pi/6}/(2\pi)$ [2]. We find

$$W\left[V_-(\eta), V_+(\eta)\right] = \frac{\mathrm{e}^{-2\mathrm{i}\pi/3}(kc)^{2/3}}{2\pi}. \quad \text{(C22)}$$

We search the two functions $c(\eta, b)$ and $s(\eta, b)$, subject to the conditions (B1), as the linear combinations

$$\begin{aligned} c(\eta, b) &= \alpha_c V_-(\eta, b) + \beta_c V_+(\eta, b) \\ s(\eta, b) &= \alpha_s V_-(\eta, b) + \beta_s V_+(\eta, b). \end{aligned} \quad \text{(C23)}$$

The determination of the coefficients in the previous linear combinations leads to

$$\alpha_c = \frac{2\pi\mathrm{e}^{2\mathrm{i}\pi/3}}{(kc)^{2/3}}V_+'(0) \qquad \beta_c = -\frac{2\pi\mathrm{e}^{2\mathrm{i}\pi/3}}{(kc)^{2/3}}V_-'(0) \quad \text{(C24)}$$

and

$$\alpha_s = -\frac{2\pi\mathrm{e}^{2\mathrm{i}\pi/3}}{(kc)^{2/3}}V_+(0) \qquad \beta_s = \frac{2\pi\mathrm{e}^{2\mathrm{i}\pi/3}}{(kc)^{2/3}}V_-(0). \quad \text{(C25)}$$

In fact, in order to exponentially improve the asymptotic expansion for the resonances of the elliptic cylinder, we only need the leading contributions

$$V_-(\eta, b) = \mathrm{e}^{-\mathrm{i}\pi/2}\zeta^{1/4}\left(b^2 - \cos^2\eta\right)^{-1/4}\mathrm{Ai}\left[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta\right]\left(1 + \underset{kc\to+\infty}{\mathrm{O}}\left(\frac{1}{kc}\right)\right) \quad \text{(C26)}$$

and

$$V_+(\eta, b) = \mathrm{e}^{-\mathrm{i}\pi/2}\zeta^{1/4}\left(b^2 - \cos^2\eta\right)^{-1/4}\mathrm{Ai}\left[(kc)^{2/3}\zeta\right]\left(1 + \underset{kc\to+\infty}{\mathrm{O}}\left(\frac{1}{kc}\right)\right). \quad \text{(C27)}$$

According to the relation $\mathrm{d}\zeta/\mathrm{d}\eta = \mathrm{e}^{-\mathrm{i}\pi/2}\zeta^{-1/2}\left(b^2 - \cos^2\eta\right)^{1/2}$, a direct differentiation of (C26) and (C27) yields

$$V_-'(\eta, b) = -\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta^{-1/4}\left(b^2 - \cos^2\eta\right)^{1/4}\mathrm{Ai}'\left(\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta\right)\left(1 + \underset{kc\to+\infty}{\mathrm{O}}\left(\frac{1}{kc}\right)\right) \quad \text{(C28)}$$

and

$$V_+'(\eta,b) = -(kc)^{2/3}\zeta^{-1/4}\left(b^2-\cos^2\eta\right)^{1/4}\mathrm{Ai}'\left((kc)^{2/3}\zeta\right)\left(1+\underset{kc\to+\infty}{\mathrm{O}}\left(\frac{1}{kc}\right)\right). \tag{C29}$$

From equations (C24) and (C25), we finally obtain

$$\begin{aligned} c(\eta,b) = {} & -2\pi\mathrm{e}^{\mathrm{i}\pi/6}\zeta(0)^{-1/4}\zeta(\eta)^{1/4}\left(b^2-1\right)^{1/4}\left(b^2-\cos^2\eta\right)^{-1/4} \\ & \times\left[\mathrm{Ai}'\left((kc)^{2/3}\zeta(0)\right)\mathrm{Ai}\left(\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(\eta)\right)\right. \\ & \left.-\mathrm{e}^{2\mathrm{i}\pi/3}\mathrm{Ai}'\left(\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(0)\right)\mathrm{Ai}\left((kc)^{2/3}\zeta(\eta)\right)\right]\left(1+\underset{kc\to+\infty}{\mathrm{O}}\left(\frac{1}{kc}\right)\right) \end{aligned} \tag{C30}$$

and

$$\begin{aligned} s(\eta,b) = {} & -2\pi\mathrm{e}^{-\mathrm{i}\pi/3}(kc)^{-2/3}\zeta(0)^{1/4}\zeta(\eta)^{1/4}\left(b^2-1\right)^{-1/4}\left(b^2-\cos^2\eta\right)^{-1/4} \\ & \times\left[\mathrm{Ai}\left((kc)^{2/3}\zeta(0)\right)\mathrm{Ai}\left(\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(\eta)\right)\right. \\ & \left.-\mathrm{Ai}\left(\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(0)\right)\mathrm{Ai}\left((kc)^{2/3}\zeta(\eta)\right)\right]\left(1+\underset{kc\to+\infty}{\mathrm{O}}\left(\frac{1}{kc}\right)\right) \end{aligned} \tag{C31}$$

as well as

$$\begin{aligned} c'(\eta,b) = {} & 2\pi\mathrm{e}^{4\mathrm{i}\pi/3}(kc)^{2/3}\zeta(0)^{-1/4}\zeta(\eta)^{-1/4}\left(b^2-1\right)^{1/4}\left(b^2-\cos^2\eta\right)^{1/4} \\ & \times\left[\mathrm{Ai}'\left((kc)^{2/3}\zeta(0)\right)\mathrm{Ai}'\left(\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(\eta)\right)\right. \\ & \left.-\mathrm{Ai}'\left(\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(0)\right)\mathrm{Ai}'\left((kc)^{2/3}\zeta(\eta)\right)\right]\left(1+\underset{kc\to+\infty}{\mathrm{O}}\left(\frac{1}{kc}\right)\right) \end{aligned} \tag{C32}$$

and

$$\begin{aligned} s'(\eta,b) = {} & 2\pi\mathrm{e}^{\mathrm{i}\pi/6}\zeta(0)^{1/4}\zeta(\eta)^{-1/4}\left(b^2-1\right)^{-1/4}\left(b^2-\cos^2\eta\right)^{1/4} \\ & \times\left[\mathrm{Ai}\left((kc)^{2/3}\zeta(0)\right)\mathrm{e}^{2\mathrm{i}\pi/3}\mathrm{Ai}'\left(\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(\eta)\right)\right. \\ & \left.-\mathrm{Ai}\left(\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(0)\right)\mathrm{Ai}'\left((kc)^{2/3}\zeta(\eta)\right)\right]\left(1+\underset{kc\to+\infty}{\mathrm{O}}\left(\frac{1}{kc}\right)\right). \end{aligned} \tag{C33}$$

Appendix D. WKB expansions for the solutions of the ordinary Mathieu equation

In order to solve asymptotically the transcendental equations $c_\ell'(\pi/2) = 0$, $s_\ell(\pi/2) = 0$, $c_\ell(\pi/2) = 0$ and $s_\ell'(\pi/2) = 0$ providing the resonances associated, respectively, with the representations A_1, A_2, B_1 and B_2, we need to construct the WKB expansions for the functions $c(\eta,b)$, $s(\eta,b)$, $c'(\eta,b)$ and $s'(\eta,b)$.

Appendix D.1. The WKB method: generalities [22]

Two linearly independent solutions of the equation

$$\frac{\mathrm{d}^2V}{\mathrm{d}z^2} = \lambda^2R^2(z)V(z) \tag{D1}$$

where λ is a large parameter, can be represented by the WKB expansions

$$V^{\pm}(z) = R^{-1/2}(z)\exp\left[\pm\lambda\int_a^z R(z)\,\mathrm{d}z\right]\sum_{n=0}^{+\infty}\frac{Y_n(z)}{(\pm\lambda)^n} \tag{D2}$$

where

$$Y_{n+1} = -\frac{1}{2R} Y_n' + \int_a^z P(z) Y_n(z)\,\mathrm{d}z + C_{n+1} \qquad \text{and} \qquad Y_0 = C_0. \tag{D3}$$

The function $P(z)$ is given by

$$P(z) = \frac{R''(z)}{4R^2(z)} - \frac{3R'^2(z)}{8R^3(z)} \tag{D4}$$

while the lower limit a and the C_n for $n \geqslant 0$ are arbitrary constants. It is important to note that any change in the arbitrary constants a and C_n corresponds to a change in the normalization of the solutions $V^{\pm}(z)$.

Appendix D.2. Application of the WKB method to the ordinary Mathieu equation

The Mathieu equation (36) can be put into the form (D1) by taking $\lambda = kc$ and

$$R(\eta, b) = \mathrm{i}\left(b^2 - \cos^2\eta\right)^{1/2} \tag{D5}$$

and can be solved by using the WKB method. $P(\eta, b)$, defined by (D4), is given here by

$$P(\eta, b) = \frac{2\left(b^2\cos^2\eta - b^2\sin^2\eta - \cos^4\eta\right) - 3\cos^2\eta\sin^2\eta}{8\mathrm{i}\left(b^2 - \cos^2\eta\right)^{5/2}}. \tag{D6}$$

By limiting the WKB expansions to the first order in $1/kc$, and by taking arbitrarily $a = 0$, $C_0 = 1$, and $C_n = 0$ for $n \geqslant 1$, we obtain

$$V^{\pm}(\eta, b) = R^{-1/2}(\eta, b)\exp\left[\pm kc\int_0^\eta R(\eta', b)\,\mathrm{d}\eta'\right]\left(1 \pm \frac{Y_1(\eta, b)}{kc} + \underset{kc\to\infty}{\mathrm{O}}\left(\frac{1}{(kc)^2}\right)\right) \tag{D7}$$

with

$$Y_1(\eta, b) = \int_0^\eta P(\eta', b)\,\mathrm{d}\eta'. \tag{D8}$$

The WKB expansions for the functions $c(\eta, b)$ and $s(\eta, b)$ which satisfy the Mathieu equation (36) and the initial conditions (B1) can then be constructed as linear combinations of the expansions $V^{+}(\eta, b)$ and $V^{-}(\eta, b)$. We find

$$\begin{aligned} c(\eta, b) =& \tfrac{1}{2} R(0, b)^{1/2} R(\eta, b)^{-1/2} \\ &\times\left\{\exp\left[kc\int_0^\eta R(\eta', b)\,\mathrm{d}\eta'\right]\left(1 + \frac{Y_1(\eta, b)}{kc} + \underset{kc\to\infty}{\mathrm{O}}\left(\frac{1}{(kc)^2}\right)\right)\right. \\ &\left.+\exp\left[-kc\int_0^\eta R(\eta', b)\,\mathrm{d}\eta'\right]\left(1 - \frac{Y_1(\eta, b)}{kc} + \underset{kc\to\infty}{\mathrm{O}}\left(\frac{1}{(kc)^2}\right)\right)\right\} \end{aligned} \tag{D9}$$

and

$$\begin{aligned} s(\eta, b) =& \frac{1}{2kc} R(0, b)^{-1/2} R(\eta, b)^{-1/2} \\ &\times\left\{\exp\left[kc\int_0^\eta R(\eta', b)\,\mathrm{d}\eta'\right]\left(1 + \frac{Y_1(\eta, b)}{kc} + \underset{kc\to\infty}{\mathrm{O}}\left(\frac{1}{(kc)^2}\right)\right)\right. \\ &\left.-\exp\left[-kc\int_0^\eta R(\eta', b)\,\mathrm{d}\eta'\right]\left(1 - \frac{Y_1(\eta, b)}{kc} + \underset{kc\to\infty}{\mathrm{O}}\left(\frac{1}{(kc)^2}\right)\right)\right\}. \end{aligned} \tag{D10}$$

We then deduce for $c'(\eta, b)$ and $s'(\eta, b)$ the following expansions:

$$c'(\eta, b) = \frac{1}{2} kc R(0, b)^{1/2} R(\eta, b)^{1/2} \times \left\{ \exp\left[kc \int_0^\eta R(\eta', b)\, d\eta'\right] \left(1 + \frac{\widetilde{Y}_1(\eta, b)}{kc} + \underset{kc\to\infty}{O}\left(\frac{1}{(kc)^2}\right)\right) - \exp\left[-kc \int_0^\eta R(\eta', b)\, d\eta'\right] \left(1 - \frac{\widetilde{Y}_1(\eta, b)}{kc} + \underset{kc\to\infty}{O}\left(\frac{1}{(kc)^2}\right)\right) \right\} \quad \text{(D11)}$$

and

$$s'(\eta, b) = \frac{1}{2} R(0, b)^{-1/2} R(\eta, b)^{1/2} \times \left\{ \exp\left[kc \int_0^\eta R(\eta', b)\, d\eta'\right] \left(1 + \frac{\widetilde{Y}_1(\eta, b)}{kc} + \underset{kc\to\infty}{O}\left(\frac{1}{(kc)^2}\right)\right) + \exp\left[-kc \int_0^\eta R(\eta', b)\, d\eta'\right] \left(1 - \frac{\widetilde{Y}_1(\eta, b)}{kc} + \underset{kc\to\infty}{O}\left(\frac{1}{(kc)^2}\right)\right) \right\} \quad \text{(D12)}$$

with

$$\widetilde{Y}_1(\eta, b) = Y_1(\eta, b) - \frac{R'(\eta, b)}{2R(\eta, b)^2}. \quad \text{(D13)}$$

References

[1] Morse P M and Feshbach H 1953 *Methods of Theoretical Physics* vols 1, 2 (New York: McGraw-Hill)
[2] Abramowitz M and Stegun I A 1965 *Handbook of Mathematical Functions* (New York: Dover)
[3] Erdelyi A 1953 *Higher Transcendental Functions* vol 3 (New York: McGraw-Hill)
[4] Uslenghi P L E and Zitron N R 1987 The elliptic cylinder *Electromagnetic and Acoustic Scattering by Simple Shapes* ed J J Bowman, T B A Senior and P L E Uslenghi (New York: Hemisphere) pp 129–80
[5] Keller J B 1956 Diffraction by a convex cylinder *IRE Trans.* **4** 312–21
[6] Keller J B 1958 A geometrical theory of diffraction *Calculus of Variations and its Applications* ed L M Graves (New York: McGraw-Hill) pp 27–52
[7] Keller J B and Levy B R 1959 Diffraction by a smooth object *Commun. Pure Appl. Math.* **12** 159–209
[8] Keller J B 1962 Geometrical theory of diffraction *J. Opt. Soc. Am.* **52** 116–30
[9] Levy B R 1960 Diffraction by an elliptic cylinder *J. Math. Mec.* **9** 147–65
[10] Smilansky U 1995 Semiclassical quantisation of chaotic billiards—a scattering approach *Mesoscopic Quantum Physics (Les Houches 1994, NATO ASI)* ed E Ackermans, G Montambaux, J-L Pichard and J Zinn-Justin (Amsterdam: North-Holland) pp 373–433
[11] Hamermesh M 1962 *Group Theory and its Application to Physical Problems* (New York: Dover)
[12] Moser P J 1982 The isolation, identification and interpretation of resonances in the radar scattering cross section for conducting bodies of finite general shape *PhD Thesis* Department of Physics, The Catholic University of America, Washington, DC
[13] Moser P J and Überall H 1983 Complex eigenfrequencies of axisymmetric, perfectly conducting bodies: radar spectroscopy *Proc. IEEE* **71** 171–2
[14] Bohigas O, Boosé D, Egydio de Carvalho R and Marvulle V 1993 Quantum tunneling and chaotic dynamics *Nucl. Phys.* A **560** 197–210
[15] Ayant Y and Arvieu R 1987 Semiclassical study of particle motion in two-dimensional and three-dimensional elliptic boxes: I *J. Phys. A: Math. Gen.* **30** 4563
[16] Waalkens H, Wiersig J and Dullin H R 1997 Elliptic quantum billiard *Ann. Phys.* **260** 50
[17] Sieber M 1997 Semiclassical transition from an elliptic to an oval billiard *J. Phys. A: Math. Gen.* **30** 4563
[18] Watson G N 1918 The diffraction of electric waves by the earth *Proc. R. Soc.* A **95** 83–99
[19] Sommerfeld A 1949 *Partial Differential Equations in Physics* (New York: Academic)
[20] Levy B R and Keller J B 1960 Diffraction by a spheroid *Can. J. Phys.* **38** 128–44
[21] Hansen E B 1962 Scalar diffraction by an infinite strip and a circular disc *J. Math. Phys.* **41** 229–45

[22] Dingle R D 1973 *Asymptotic Expansions: their Derivation and Interpretation* (London: Academic)
[23] Segur H, Tanveer S and Levine H (ed) 1991 *Asymptotics Beyond all Orders* (New York: Plenum)
[24] Berry M V and Howls C J 1990 Hyperasymptotics *Proc. R. Soc.* A **430** 653–68
[25] Olver F W J 1954 The asymptotic solutions of linear differential equations of the second order for large values of a parameter *Phil. Trans. R. Soc.* A **247** 307–27
[26] Franz W 1957 *Theorie der Beugung Elektromagnetischer Wellen* (Berlin: Springer)
[27] Streifer W and Kodis R D 1964 On the solution of a transcendental equation arising in the theory of scattering by a dielectric cylinder *Q. Appl. Math.* **21** 285–98
[28] Olver F W J 1954 The asymptotic expansion of Bessel functions of large order *Phil. Trans. R. Soc.* A **247** 328–68
[29] Fock V A 1965 *Electromagnetic Diffraction and Propagation Problems* (Oxford: Pergamon)
[30] Bouche D, Molinet F and Mittra R 1996 *Asymptotic Methods in Electromagnetics* (Berlin: Springer)
[31] Pathak P H 1979 An asymptotic analysis of the scattering of plane waves by a smooth convex cylinder *Radio Sci.* **14** 419–35
[32] Wolfram S 1996 *The Mathematica Book* (Cambridge: Cambridge University Press)